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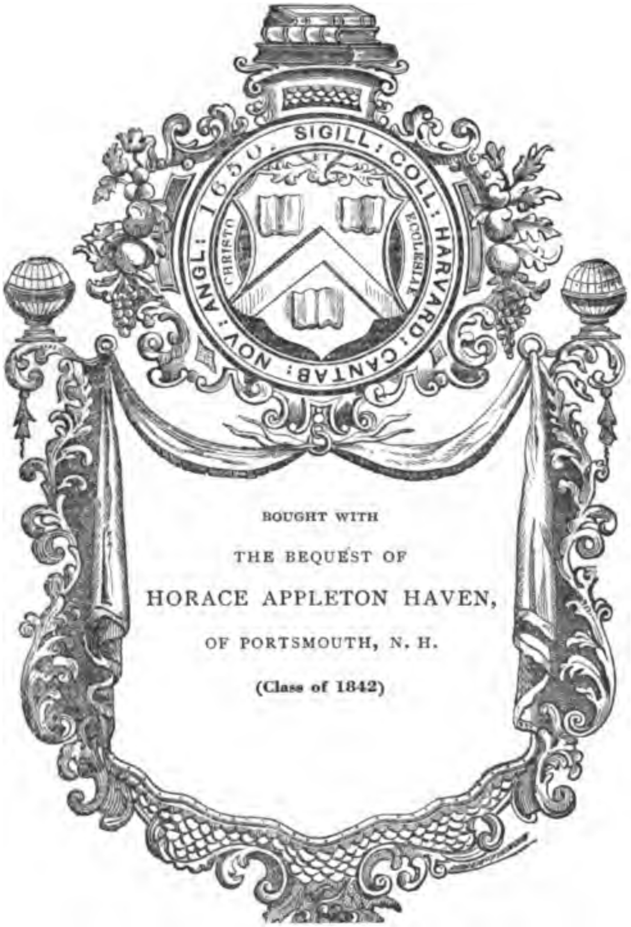

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FIRST COURSE IN CALCULUS

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AND

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PREFACE

In presenting this book to the public the authors have had in mind primarily the needs of their own students in mathematics; they hope, however, that the book may be found of service to other colleges and technical schools.

In the development of the subject the method of limits has been used exclusively. A discussion of the various orders of infinitesimals is thereby avoided, and the student is made familiar with the use of limits as the basis of a rigorous demonstration, a drill that will be of service to him in his subsequent study of mathematics.

The book has not been divided into Differential Calculus and Integral Calculus. The arrangement adopted has certain distinct advantages. The student becomes familiar with integration as an inverse operation as soon as he has learned to differentiate, thus making it possible to introduce early a broader field of simple applications of the calculus. Furthermore, by this arrangement, it is possible to postpone until late in the course a consideration of some of the more difficult topics; thus the expansion of functions, singularities of plane curves, etc., are postponed to such a time as the student's knowledge of the fundamental principles of calculus makes it possible for him more fully to appreciate a more rigorous treatment of the subject.

Throughout the book an attempt has been made to select such problems and applications as arise in the actual experience of an engineer without introducing technical difficulties beyond the experience of the average sophomore student who has had the usual course in high school physics. The book has not been written, however solely from the point of view of the engineer. The applications are such as the general student will find both helpful and stimulating in showing the broad use of calculus in practical problems.

More attention is given to the elementary applications to mechanics than is usual in calculus and perhaps less to geometry, it being the thought of the authors that the two should stand in about the same relative importance.

The miscellaneous problems at the close of each chapter afford a general review of the calculus up to that point. Some of these problems will doubtless be found sufficiently difficult to test the mathematical ability of the strongest students.

Another feature of the arrangement worthy of mention is the use of the spiral method so commonly employed in elementary mathematical texts. For example, when the subject of maxima and minima is first discussed, only a geometric proof is given. Subsequently a more formal proof is introduced with additional exercises of a more difficult character. The same statement applies to other important topics.

The addition of a carefully prepared index makes the book readily available for reference purposes.

The authors take this opportunity of expressing their appreciation of the assistance and helpful suggestions that their colleagues in the mathematical and engineering departments at the University of Illinois have given them during the preparation of the book. They wish to acknowledge their obligations to Professor Alexander Ziwet of the University of Michigan, to Professor E. W. Davis of the University of Nebraska, and to Professor A. H. Wilson of the Alabama Polytechnic Institute, who have read the manuscript; also to Professor C. N. Haskins, Professor A. G. Hall, and Dr. E. L. Dodd, who have rendered valuable assistance in seeing the book through the press.

E. J. Townsend.
G. A. Goodenough.

University of Illinois
January, 1908

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FIRST COURSE IN CALCULUS

GREEK ALPHABET

Letters.		**Names.**	**Letters.**		**Names.**
Capitals.	**Lower Case.**		**Capitals.**	**Lower Case.**	
Α	α	Alpha	Ν	ν	Nu
Β	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	Ο	ο	Omicron
Δ	δ	Delta	Π	π	Pi
Ε	ε	Epsilon	Ρ	ρ	Rho
Ζ	ζ	Zeta	Σ	σ ς	Sigma
Η	η	Eta	Τ	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
Ι	ι	Iota	Φ	φ	Phi
Κ	κ	Kappa	Χ	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
Μ	μ	Mu	Ω	ω	Omega

A FIRST COURSE IN CALCULUS

CHAPTER I

CONSTANTS, VARIABLES, FUNCTIONS

1. Constants, variables. In the applications of mathematics to physical problems, we meet with such magnitudes as velocity, force, mass, length, area, volume, etc. The measure of a magnitude is expressed by means of a number, represented by a figure or letter, which denotes the ratio that the given magnitude bears to some standard magnitude of the same kind adopted as a unit. For the purposes of calculation, it is the number which is of fundamental importance. For the sake of brevity, however, we shall often speak of "a velocity v" or "an area A," etc., instead of using the longer but perhaps more precise expressions "a velocity whose measure is v units" or "an area whose ratio to the standard unit of area is A," etc.

In any particular discussion, the magnitudes and consequently the corresponding numbers may or may not change. A number which remains unchanged is called a *constant*. A symbol is then said to represent a constant when it denotes but one value in a given discussion. A symbol satisfying this condition is itself often called a constant.

When a number is permitted to assume different values in the same discussion, it is called a *variable*. A symbol is said to represent a variable if it has assigned to it different values in the same discussion. Here again it is usual to speak of the symbol itself as the variable.

For example, suppose a body falls from rest. The law which gives the relation between the time t and the distance s through which the body falls is expressed by the equation

$$s = \tfrac{1}{2} gt^2. \qquad (1)$$

Here the number g is a constant denoting the acceleration at a point on the earth's surface. On the other hand, the time t and the distance s are variables, whose mutual relation is determined by the given equation.

Again, in the equation of the circle

$$(x - 2)^2 + (y - 3)^2 = r^2, \tag{2}$$

2 and 3 are constants which determine the center of the circle, and r is a constant denoting the length of the radius However, x and y are variables whose corresponding values, as determined by the given equation, fix the various positions of the generating point. To the constant r we may assign any value at pleasure, but when it has been once assigned it must remain unchanged during the investigation. A constant of this character is called an *arbitrary constant.*

It is often desirable to indicate the numerical value of an expression, that is, its value without regard to its algebraic sign. We do this by writing a vertical line before and after the expression; thus, $|a|$, $|x + y|$ denote the numerical values of a and $x + y$, and are equal to $|-a|$, $|-x - y|$, respectively. $|a|$ is read, "the absolute value of" a.

2. Functions. In each of the illustrations given in the preceding article, it will be observed that the variables involved stand in an intimate relation to each other. For example, in the law of falling bodies, s and t are so related that to any value assigned to t there corresponds a definite value of s. Moreover, to every positive value that we may give s, the given relation between s and t determines corresponding values of t. Again, in the equation of the circle the variables x and y are related in a similar manner. Other illustrations of such relations between variables are familiar to the student from his study of elementary mathematics and physics. Whenever such a relation exists between two variables, we say that the variables are connected by a functional relation, or that the one is a function of the other. We may then define a function as follows:

If two variables are so related that for each value which may be assigned to the one there are determined one or more definite values of the other, the second variable is said to be a *function* of the first.

The variable to which we may assign arbitrarily chosen values is called the *independent variable.* It is also frequently referred to simply as the variable or argument. The variable which is thus determined, that is, the function, is sometimes called the *dependent variable.*

In the illustrations given in Art. 1, s in (1) is a function of t, and in (2) y is a function of x, the independent variables being t and x respectively. A function has a meaning, that is, it is defined, for only those values of the independent variable that give rise to definite values of the function. It may happen that the analytic expression of the relation between the function and the independent variable defines that function for certain values only of the independent variable. For example, in the equation of the circle $x^2 + y^2 = r^2$, the values of y become imaginary for x less than $-r$ or greater than r. The function y is therefore defined only for $-r \leqq x \leqq r$, or as we say for the interval $(-r, r)$. Again, the function may be defined for certain kinds of values only, as positive values, or integral values. The essential characteristic of a function is that to each value that the independent variable is permitted to take by the conditions of the problem, there corresponds a definite value or definite values of the dependent variable or function.

Given the analytic expression of the relation between two variables, it is generally a matter of indifference which variable we select as the independent variable. In any particular problem we assume that variable as the independent variable which best suits our convenience and the nature of the problem. For example, a railroad train is moving at a constant speed. There exists a functional relation between the time and the distance traversed. If the time at which the train passes certain stations is desired, the distance should be taken as the independent variable; if the distance between stations is the subject of inquiry, then the time becomes the independent variable.

A function may depend for its value upon two or more independent variables. For example, the area of an ellipse is a function of the major and the minor axes; the volume of a gas is a function of the temperature and of the pressure to which it is subjected. Such functions are said to be functions of two variables. For the present we shall consider only functions of a single variable; later, some of the properties of functions of two or more variables will be discussed. As we proceed, the student will observe that the calculus is in reality a branch of the study of functions.

3. Notation. Even when the algebraic relation of the independent and dependent variables is known, it is often convenient to introduce a general symbol that shall stand for the function as expressed in terms of its variable. This is done by inclosing the variable in a parenthesis and prefixing some letter to it. Thus, a function of x may be denoted by such symbols as the following:

$$f(x),\ F(x),\ \phi(x),\ \psi(x), \text{ etc.}$$

It is to be understood that the letters f, F, ϕ, ψ, etc., are symbols of functionality and not factors. Whenever two or more different functions are employed in the same discussion, a different symbol must be used for each.

If a function depends upon two variables instead of one, it may be expressed symbolically by $f(x, y)$, $\phi(x, y)$, etc.

When, in connection with any discussion, we have once defined $f(x)$, then $f(a)$ denotes the same expression with x replaced by a. In a similar way, we may form functions $f(t)$, $f(4)$, $f(x+h)$, etc. Thus, if we have

$$f(x) = 3x^2 + 7x + 9,$$

it follows that

$$\begin{aligned} f(t) &= 3t^2 + 7t + 9, \\ f(4) &= 3 \cdot 4^2 + 7 \cdot 4 + 9 = 85, \\ f(x+h) &= 3(x+h)^2 + 7(x+h) + 9. \end{aligned}$$

Instead of replacing the variable by another variable or by a constant, we may replace it by any function of the same variable or of a different variable. Thus we may write

$$f[\phi(x)],\ F[f(y)],\ \psi(\sin\theta), \text{ etc.}$$

Given, for example, the function

$$F(x) = x^2 + 3x + 7,$$

we have

$$F(\sin x) = \sin^2 x + 3\sin x + 7.$$

EXERCISES.

1. Given $f(x) = x^3 - 6x^2 + 5x - 14$. Find the values of $f(4)$, $f(0)$, $f(-3)$. Write the expression for $f(\sin\theta)$ and for $f(x+h)$.

2. If $\phi(x) = 5x^2 - 2x + 8$, write expressions for $\phi(x^2)$, $\phi(-x^3)$, $\phi(\tan\theta)$, $\phi(0)$, $\phi(t)$.

3. If $F(\theta) = \cos\theta$, find the value of $F(0), F\left(\frac{\pi}{4}\right), F\left(\frac{\pi}{2}\right), F(\pi)$.

4. If $f(x) = x^5 - 4x^3 + x$, show that $f(-x) = -f(x)$.

5. If $f(x) = x^4 - 6x^2 + 1$, show that $f(-x) = f(x)$.

6. Given $F(x) = 3 - \sqrt{x}$ and $f(y) = y^2 + 4$. Find $F[f(y)]$ and $f[F(x)]$, also $f[f(y)]$ and $F[F(x)]$.

7. Given $y = f(x) = \dfrac{1+x}{1-x}$, and $z = f(y) = f[f(x)]$; express z as a function of x.

8. If $\phi(x) = 2x^2 - 1$, show that $\phi(\cos\theta) = \cos 2\theta$.

9. If $\phi(x) = 2x\sqrt{1-x^2}$, show that $\phi(\sin\theta) = \phi(\cos\theta) = \sin 2\theta$.

10. If $f(x) = \log_a x$, show that $f(x) - f(y) = f\left(\frac{x}{y}\right)$, and $f(x) + f(y) = f(xy)$.

11. If $f(\theta) = \cos\theta$, show that $f(\theta) = f(-\theta) = -f(\pi + \theta) = -f(\pi - \theta)$.

12. If $f(x) = \sin x$ and $\phi(x) = \cos x$, find $f(x)\cdot\phi(y) \pm \phi(x)\cdot f(y)$.

13. If $f(x) = e^x$,* show that $f(x)\cdot f(y) = f(x+y)$.

14. If $f(x) = \sqrt{1-x^2}$, find $f(\sin\theta)$ and $f(\cos\theta)$.

15. If $f(x) = \sqrt{1+x^2}$, find $f(\tan\theta)$.

16. If $f(x) = \tan x$, find $\dfrac{f(x) - f(y)}{1 + f(x)\,f(y)}$.

17. The following are laws found in elementary physics. In each case having selected the independent variables, and the function, write the laws (1) in functional notation, and (2) directly in terms of the variables.

(a) The time of oscillation of a pendulum is proportional to the square root of the length of the pendulum.

(b) The kinetic energy of a moving body is proportional to the square of the velocity.

(c) An electric current is proportional to the electromotive force and inversely proportional to the resistance of the circuit.

(d) Heat generated in a circuit by a current is proportional to the square of the current and to the resistance of the circuit.

4. Graphs of functions. In accordance with the principles of analytic geometry, we may give a geometric interpretation to the relation of a function to its variable. If we lay off the values of the independent variable as abscissas, the corresponding values of the function may be used as ordinates. The assemblage of the extremities of these ordinates is called the *graph* of the function. It is to be noted that the ordinates of the points on the graph and not the graph itself represent the values of the function.

---

* e is used to denote the base of the Napierian system of logarithms, namely, 2.718 . . . Log x indicates the logarithm of x to the Napierian base e. When any other base is used, that fact will be indicated by a subscript, as $\log_a x$, $\log_{10} 30$.

It is assumed that the student is already familiar with the graphical representation of functions from his study of analytic geometry. However, a few exercises are introduced by way of review. Some of the functions given are of frequent use in physics and mechanics. In this work it is well to use coördinate or cross-section paper.

Draw the graphs of the two functions,

$$s = \tfrac{1}{2} gt^2 \tag{1}$$

and

$$v = gt. \tag{2}$$

The first, as has been stated, gives the relation between the space passed over by a body falling from rest and the time t. As we shall see later, the second gives the velocity of the body at any instant t. The time is taken as the independent variable, and the values of t are therefore laid off on the axis of abscissas to some convenient scale. In the first equation the values of s and in the second the values of v are laid off as ordinates. The scale used for the ordinates need not necessarily be the same as that used for the abscissas. The graph of (1) is a parabola, Fig. 1, passing through the origin, and that of (2) is a straight line, Fig. 2, also passing through the origin.

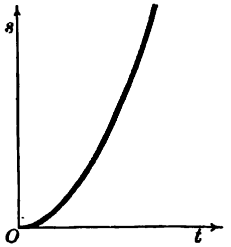

Fig. 1.

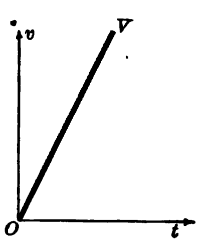

Fig. 2.

The line OV, Fig. 2, is a speed curve with the time as the independent variable, or, as it is usually expressed, a speed curve on a time base. In certain investigations, it is desirable to use the space or distance traversed as the variable. To obtain the speed curve on a space or distance base in the case of a falling body, we eliminate t between equations (1) and (2). The resulting equation,

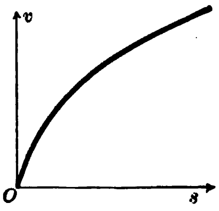

Fig. 3.

$$v = \sqrt{2 gs}, \tag{3}$$

expresses the speed v as a function of the space s. The graph is evidently a parabola, Fig. 3, with its axis coincident with the s-axis.

EXERCISES.

1. Draw the graphs of the following functions:

(a) $3x - 7$; (b) $\sqrt{x} + 5$; (c) $x^3 - x + 3$; (d) $\frac{x-5}{x+7}$.

2. Draw the graphs of the functions:

(a) 10^x; (b) $\log_{10}(1 + x)$; (c) $10^x + 10^{-x}$.

3. Plot the equations:

(a) $y = \sin\left(x - \frac{\pi}{6}\right)$; (b) $y = \cos 3x$.

(c) $y = 2 + \sin\left(x - \frac{\pi}{6}\right) - 4\cos 3x$.

4. The relation between temperatures on the Fahrenheit and centigrade scales is given by the equation

$$y = \tfrac{9}{5}x + 32.$$

Taking values of x (temperature C.) as abscissas, draw the graph of this function.

5. The efficiency E of a hoisting device depends upon the load L raised, and the general form of the equation expressing this dependence is

$$E = \frac{L}{aL + b},$$

where a and b are constants. Draw the graph of this relation, using the following values: $a = 3.2$; $b = 36$. Assume that L varies from 50 to 300 lb.

6. The formula:

$$H = \frac{v}{825}\left(T - \frac{wv^2}{g}\right)$$

gives the horsepower H transmitted by a rope running at a speed v. T denotes the tension in the tight side of the rope, and w the weight of rope per foot of length. Take $T = 400$, $w = 0.7$, $g = 32$, and draw a graph showing the variation of horsepower with the rope speed. Take values up to $v = 120$.

5. Classification of functions. In discussing functions, it is often convenient to divide them into classes according to the properties that they possess or the character of the operations that give rise to them. Among these subdivisions are the following:

(a) *Single-valued functions, multiple-valued functions.* The student may have noticed in the illustration of falling bodies that the value of t determines only one value of s, while if s is the independent variable, each value assigned to it determines two values of t. To make this distinction clear, we say that s is a single-valued function of t, but t is a multiple-valued function of s.

A function is said to be *single-valued* if to each value of the independent variable there corresponds but one value of the function. If more than one value of the function corresponds to each value of the variable, we call the function *multiple-valued.*

(b) *Explicit functions, implicit functions.* If a function is expressed directly in terms of its variable, we say that it is an *explicit function* of that variable. Thus

$$y = 3x^3 + 7x + 4 \tag{1}$$

is such a function.

If, on the other hand, the dependent variable is not expressed directly in terms of the independent variable, then the former is said to be an *implicit function* of the latter. Thus in

$$4xy + 8y - 3x^2 = 7 \tag{2}$$

y is an implicit function of x. When, however, we solve this equation for y, we have

$$y = \frac{3x^2 + 7}{4x + 8}, \tag{3}$$

and y has now become an explicit function of x.

The relation of the two variables may be given by an equation either of the form

$$f(x, y) = 0 \tag{4}$$

or

$$y = f(x). \tag{5}$$

Either variable is a function of the other. In (4) either variable is an implicit function of the other. In (5) y is an explicit function of x, but x is an implicit function of y.

(c) *Rational functions, irrational functions.* If a function can be written in the form

$$y = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n, \tag{6}$$

where the a's are constants and n is a positive integer, we say that y is an *integral rational function* of x. In algebra, we called such functions polynomials.

A *rational function* is the quotient of two integral rational functions or polynomials. It may therefore be written in the form

$$y = \frac{a_0 + a_1x + a_2x^2 + \dots + a_nx^n}{b_0 + b_1x + b_2x^2 + \dots + b_mx^m}, \tag{7}$$

where the a's and the b's are constants and m, n are positive integers. Functions which are not rational are called *irrational.*

In (1) and (3) y is a rational function of x. In each of the following relations y is an irrational function of x.

$$y = \pm \sqrt{x^2 + 4}. \tag{8}$$

$$y = x^2 - bx^{\frac{1}{2}} + c. \tag{9}$$

$$x^3 + y^3 = 1. \tag{10}$$

(d) *Algebraic functions, transcendental functions.* We may also classify functions according to the number of times we need to apply the fundamental operations, addition, subtraction, multiplication, division, raising to a power, and the extraction of a root to known constants and the independent and dependent variables in expressing the required functional relation. When these fundamental operations need be applied only a finite number of times, we say that the function is *algebraic.* Such a function satisfies an algebraic equation; that is, y is an algebraic function of x if the relation between x and y is given by the equation

$$f_0(x)\, y^n + f_1(x)\, y^{n-1} + \ldots + f_n(x) = 0,$$

where $f_0(x),\ f_1(x), \ldots f_n(x)$ are polynomials in x.

By comparison with (c) it will be seen that rational functions are also algebraic. A rational function is, however, an explicit algebraic function. While algebraic functions may be and often are expressed in terms of the independent variable by means of infinite processes, it is characteristic of explicit algebraic functions that they can always be so expressed without aid of infinite processes.

Functions which are not algebraic are called *transcendental.* Such functions include trigonometric functions, exponential functions, logarithmic functions, hyperbolic functions, etc. Other transcendental functions are studied in the higher branches of mathematics.

(e) *Inverse functions.* If we have given y as an explicit function of x, say

$$y = x^2 + 4, \tag{11}$$

x is also a function of y. We can find the explicit expression of this function by solving the above equation for x in terms of y. This gives

$$x = \pm \sqrt{y - 4}. \tag{12}$$

We call the two functions $x^2 + 4$ and $\pm \sqrt{y - 4}$ inverse functions.

If we have given

$$y = f(x),$$

so far as the functions to be considered in the present volume are concerned, we can usually solve this equation for x, thus obtaining

$x = \phi(y)$; then the functions $f(x)$ and $\phi(y)$ are *inverse functions.* The student has already met such functions in algebra and in trigonometry. For example, sin x and arc sin y,* log x and e^y, x^n and $y^{\frac{1}{n}}$, etc., are such functions.

It should be noted that the inverse of a single-valued function is often multiple valued. Equations (11) and (12) furnish an example. The same is true of the inverse trigonometric functions.

EXERCISES.

1. $F(x, y) = axy + bx + cy + d = 0$. Express y as an explicit function of x; also x as an explicit function of y.

2. Given arc sin x + arc sin $y = \beta$; express y as an explicit function of x.

3. Given $pv^m = C$, in which m and C are constants; express v as an explicit function of p.

4. Given $xe^{-\frac{y}{4}} = C$; express y as an explicit function of x.

5. Given
$$x = f(\phi) = a(\phi - \sin\phi),$$
$$y = F(\phi) = a(1 - \cos\phi);$$
express x as an explicit function of y.

6. Given $u^2 - 2mu + v^2 = 0$; make u an explicit function of v.

7. Given $\log p = A + \frac{B}{T} + \frac{C}{T^2}$; express T as an explicit function of p.

8. Find the functions inverse to the following functions:

(a) x^3; (b) $\sqrt{x^2 - a^2}$; (c) $\log(x^2 + a^2)$.

9. If $f(x) = \log(x + \sqrt{x^2 + 1})$, find the inverse function.

10. If $F(u) = \frac{1}{2}\log\frac{1 + u}{1 - u}$, find the inverse function.

11. If $f(x) = ma^x$ and $F(x) = \log_a x$, for what value of m are the functions inverse?

12. If $y = f(x) = \frac{x - 3}{2x + 5}$, find the inverse function.

13. Given $y = f(x) = \frac{ax + b}{cx - a}$. If $x = \phi(y)$ is the inverse function, show that $f(x) = \phi(x)$.

14. If $f(x)$ and $\phi(x)$ are inverse functions, show that the graphs of these functions are symmetrical with respect to a line bisecting the angle between the positive X- and Y-axes. Test this relation by drawing the graphs of the inverse functions e^x and log x.

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* Arc sin y, arc tan y, etc., are frequently written $\sin^{-1} y$, $\tan^{-1} y$ in American and English texts. x = arc sin y is read "x equals the arc (or angle) whose sine is y."

CHAPTER II

THEORY OF LIMITS

6. Introductory problems. The student is already somewhat familiar with the use of limits from his study of elementary mathematics. In geometry the area of a circle is defined as the limit of the area of a regular inscribed or circumscribed polygon; in analytic geometry the equation of the tangent to a curve is found by considering the tangent as the limit of a secant; again, in algebra the limit of the sum of a finite number of terms of an infinite series is discussed. The notion of the limit is most fundamental and important in the study of calculus. It is essential therefore that the student should gain a clear conception of the limit and of the principles involved in its definition and use.

To emphasize the important rôle that limits play in a certain class of problems, the solution of which is the peculiar province of the calculus, we introduce here a few illustrative examples.

(1) Required the angle that the tangent to a given curve $y = f(x)$ at the point (x_1, y_1) makes with the X-axis. Through the given point P_1 we draw a secant P_1P cutting the curve in a second point P, whose coördinates are x, y. If θ is the angle which this secant makes with the X-axis, we have

$$\tan\theta = \frac{y - y_1}{x - x_1}.$$

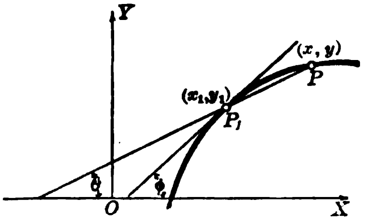

Fig. 4.

As the point P is made to approach the given point P_1, the secant line becomes in its limiting position the tangent to the curve at P_1. Hence angle θ has angle ϕ as its limit, and consequently the limiting value of the fraction $\frac{y - y_1}{x - x_1}$ gives the value of $\tan\phi$.

(2) Required the speed of a moving point at any instant. Suppose that the point is moving in a straight line OX, Fig. 5, and

that the speed is required at the instant the point passes P_1. Let s_1 denote the distance of P_1 from some chosen origin O, and t_1 the time taken to traverse the distance s_1. We choose a second point P on OX, denote its distance from O by s, and the time required to move through the distance s by t. Now the number which denotes the average speed between P_1 and P is the ratio of the number of units of length in P_1P to the number of units of time occupied in passing from P_1 to P. That is,

$$\text{average speed between } P_1 \text{ and } P = \frac{s - s_1}{t - t_1}.$$

Fig. 5.

This average speed is not, however, necessarily the speed at the instant t_1. It would be if the speed were constant throughout the interval, but, in general, not otherwise. If the time interval $t - t_1$ is taken shorter, the average speed is a closer approximation to the speed at P_1; if it is taken still shorter the approximation is still closer. Hence, to arrive at the speed at the instant t_1, we must take the limiting value of the quotient $\frac{s - s_1}{t - t_1}$ as the interval $t - t_1$ becomes indefinitely small. When the motion of the point follows some simple law, the speed at a given instant can be found by elementary methods. For example, in the case of a body falling in vacuo, the relation between distance and time is expressed by the equation

$$s = \tfrac{1}{2} gt^2.$$

For the time t_1, the distance is

$$s_1 = \tfrac{1}{2} gt_1^2,$$

whence
$$\frac{s - s_1}{t - t_1} = \tfrac{1}{2} g \frac{t^2 - t_1^2}{t - t_1} = \tfrac{1}{2} g (t + t_1),$$

which has the limiting value gt_1 as t approaches t_1. The speed at the time t_1 is therefore gt_1. For a less simple law of motion, this algebraic method of determining the speed might be difficult of application. The calculus furnishes a means of finding the limit directly from the law of motion with comparative ease, however complicated the law may be.

(3) Required the area between a curve $y = f(x)$, the X-axis and two given ordinates. Let $AEFB$, Fig. 6, be the figure whose

area is required. We divide the given area into strips as indicated, and construct on each of the dividing ordinates a rectangle. The area of one of these rectangles is

$$(x_{k+1} - x_k) \cdot f(x_k),$$

and the sum of the areas of all the rectangles may be denoted by

$$\sum (x_{k+1} - x_k) \cdot f(x_k).$$

It will be shown in Art. 82 that as the number of rectangles is indefinitely increased and the width of each is correspondingly diminished, the sum of the areas of the rectangles approaches a limit which we call the area of the figure. Hence the solution of this problem likewise reduces to the determination of a limit.

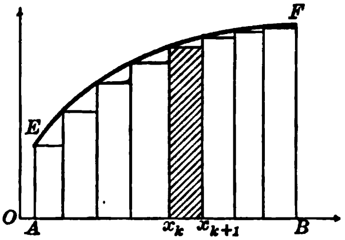

Fig. 6.

The preceding examples are sufficient to show the importance of limits in a large class of problems. In the following paragraphs we shall discuss somewhat fully the exact definition of a limit and the laws that govern operations with limits.

7. Definition of a limit. The general notion of a limit is perhaps fairly clear from the illustrative problems already given. We must, however, formulate a definition with sufficient accuracy that it may be made the basis of a mathematical investigation. For this purpose, let us consider two variables, one a function of the other; say, $y = f(x)$.

Let the independent variable x vary in such a manner that it differs less and less numerically from some constant a, that is, in such a way that $|x-a|$ becomes and remains less than any positive constant however small. We assume also that x does not become equal to a. We say then that x approaches a, and indicate that fact by writing $x \doteq a$, which is to be read "x approaches a."

As the independent variable x approaches the constant a the dependent variable or function $f(x)$ assumes a corresponding set of values. When x becomes very nearly equal to a, these values of $f(x)$ may at the same time become very nearly equal to some constant, say A. Moreover, it may occur that we may not only make $f(x)$ differ as little as we please from A by taking a value of x sufficiently near to a, but that it will remain at least as close to A

for all values of x which lie between a and this chosen value of x. When these conditions are fulfilled, we call A the *limit* of the function $f(x)$ as x approaches a.

The problem of falling bodies, Art. 6, furnishes a good illustration. The function $\frac{1}{2}g(t + t_1)$ not only may be made to differ as little as we please from gt_1 by taking t sufficiently near to t_1, but as t assumes any value still closer to t_1 the function differs still less from gt_1. Hence we were justified in speaking of gt_1 as the limit of $\frac{1}{2}g\,(t + t_1)$, as $t \doteq t_1$.

We may now define a limit as follows:

If a constant A can be found such that, as x approaches a, $f(x) - A$ becomes and remains less numerically than any constant however small, then A is called the limit of $f(x)$ as x approaches a.

This definition may also be stated somewhat more formally as follows:

Given any positive constant σ as small as we please to make it, if another positive constant h can be found such that $f(x)$ differs from some constant A by less than σ for all values of x differing numerically from a by less than h ($x = a$ not included), then we say that $f(x)$ has the limit A as x approaches a.

If the function $f(x)$ has the limit A we indicate that fact by writing

$$\underset{x \doteq a}{L} f(x) = A. \qquad (1)$$

Since by the definition of a limit $f(x)$ can be made to differ from A as little as we please by the proper selection of x, we may also write

$$f(x) = A + \epsilon(x), \qquad (2)$$

where $\epsilon(x)$ can be made as small as we choose by taking x sufficiently near to a. Equations (1) and (2) are equivalent and are merely different forms of expression for the same relation. In some demonstrations form (2) will be found the more convenient.

Ex. Consider the limit of the function

$$f(x) = \frac{4}{x^2 + 1}$$

as $x \doteq 2$. By taking the value of x sufficiently near 2, we may make the value of $f(x)$ differ as little as we please from $\frac{4}{5}$. Hence we have

$$\underset{x \doteq 2}{L} \frac{4}{x^2 + 1} = \frac{4}{5}.$$

Geometrically, the definition of a limit implies that $f(x)$ has the limit A if, having given an arbitrarily small strip parallel to the X-axis and including $y = A$, it is possible to draw another strip parallel to the Y-axis and including $x = a$ such that for all values of x lying within the last strip the corresponding values of $f(x)$ fall also within the first strip.

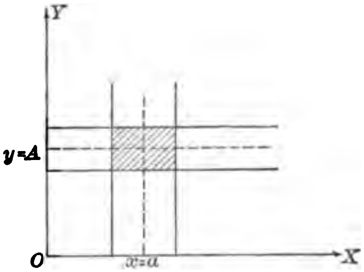

Fig. 7.

The student may verify this geometric interpretation for the case of falling bodies ($s = \frac{1}{2}gt^2$) by assuming $t_1 = 2$. Notice that as the s strip is taken smaller and smaller, the corresponding t strip in this case also becomes smaller.

8. In the preceding discussion of limits nothing was said about the value of $f(x)$ for $x = a$. The limit depends for its value upon the values of $f(x)$ for x very close to $x = a$, or, as we frequently say, "in the neighborhood of a," and it is not affected by the value which $f(x)$ takes for $x = a$. The value which the function takes for $x = a$, that is, $f(a)$, as we shall see later, may or may not be equal to the limit of $f(x)$ as x approaches a.

To determine whether a function has a limit as the variable approaches a definite value, we must consider all values of the function in the neighborhood of the limiting value of the variable. In other words, a limit of the values of the function must exist as the variable approaches its limit from either direction, and these two limiting values of the function must be equal. The following example serves as an illustration.

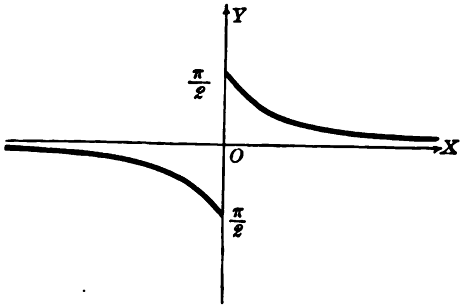

Fig. 8.

Ex. Given the function $y = \text{arc tan}\,\frac{1}{x}$, the graph of which is shown in Fig. 8. When the independent variable is restricted to positive values, we obtain the limit $\frac{\pi}{2}$ as $x \doteq 0$; when negative values only are considered, we get $-\frac{\pi}{2}$ as $x \doteq 0$. Hence the function cannot be said to have a limit as $x \doteq 0$.

9. Infinity. In the discussion of limits thus far, we have considered only the case where the independent variable approaches a definite number. The student is familiar from his study of algebra and geometry also with the case where the independent variable increases without limit. For example, the area of a circle is the limit of the area of a regular inscribed polygon as the number of sides increases without limit. Again, we obtain the value of an infinite series by taking the limit of the sum of a finite number of terms as the number of terms increases without limit. We say in such cases that the independent variable *becomes infinite,* and express that fact by writing $x = \infty$, $n = \infty$, etc. Here the symbol = is not to be understood in the ordinary sense of "equals," but rather in the sense of "becomes," and the above expressions should be read "x becomes infinite," "n becomes infinite," etc. Instead of the independent variable, the function may become infinite. This may occur when the variable also becomes infinite, or when it approaches a definite number. Thus $\frac{1}{x}$ becomes infinite as $x \doteq 0$. While it is customary to write

$$\underset{x \doteq 0}{L} \frac{1}{x} = \infty,$$

the function cannot be said to have a limit, because it does not approach any *definite* number however large that number may be.

When a function has the limit zero, it is very often spoken of as an *infinitesimal.*

The introduction of the concept infinity extends the use of limits so as to include a large number of valuable applications, of which the following are illustrations.

Ex. 1. An air pump is used to remove air from an inclosed space. At each stroke of the piston, part of the air is removed, and in consequence the density of the air remaining is diminished. Let d denote the original density, d_n the density after n strokes of the piston, v the volume of the inclosed space, and v' the volume of the pump cylinder. From physics, we have

$$d_n = d\left(\frac{v}{v + v'}\right)^n = \frac{d}{\left(1 + \frac{v'}{v}\right)^n}.$$

Here the density d_n is a function of the number of strokes. If n is increased without limit, we have, since the denominator is greater than unity,

$$\underset{n=\infty}{L}\frac{d}{\left(1+\frac{v'}{v}\right)^n}=0.$$

Hence, as the variable increases without limit, the function approaches 0 as a limit. With a perfect pump we may therefore make the density as small as we please by taking the number of strokes sufficiently large.

Ex. 2. The work done by a gas in expanding from a volume v_1 to a volume v according to the law

$$pv^k = \text{const.}$$

is given by the expression

$$\frac{p_1v_1^k}{k-1}\left[\frac{1}{v_1^{k-1}}-\frac{1}{v^{k-1}}\right].$$

If the expansion proceeds without limit, that is, if $v = \infty$, the expression for the work becomes

$$\underset{v=\infty}{L}\frac{p_1v_1^k}{k-1}\left[\frac{1}{v_1^{k-1}}-\frac{1}{v^{k-1}}\right]=\frac{p_1v_1^k}{k-1}\left[\frac{1}{v_1^{k-1}}-0\right]=\frac{p_1v_1}{k-1}.$$

In this case the function approaches a definite limit as the variable increases without limit.

Ex. 3. If the gas expands according to Boyle's law,

$$pv = \text{const.},$$

the expression for the work done is

$$p_1v_1\log\frac{v}{v_1}.$$

If in this case the volume v increases without limit, we have for the work done

$$\underset{v=\infty}{L}p_1v_1(\log v-\log v_1)=\infty;$$

that is, the function also increases without limit.

Ex. 4. In the preceding examples, suppose the gas to be compressed, so that $v < v_1$. The work required for compression is given by the expression for the work of expansion with its sign changed. If now the volume v is made to approach 0, we have for the two cases, respectively,

$$\underset{v\doteq 0}{L}\frac{p_1v_1^k}{k-1}\left[\frac{1}{v^{k-1}}-\frac{1}{v_1^{k-1}}\right]=\infty,$$

and

$$\underset{v\doteq 0}{L}p_1v_1\log\frac{v_1}{v}=\infty.$$

Hence, in either case the work required to compress the gas into zero space is infinite. Here we have an example in which the function becomes infinite.

10. Continuity of functions. We have seen that the value of a limit depends upon the values of the function in the vicinity of the limiting value of the variable, and not upon the value of the function at that point. In some cases the limit is the same as the value of the function at the limiting point, and in some cases it is not. If the two values are equal, we say that the function is *continuous* at the point in question. If, on the other hand, the limit is different from the value of the function at the limiting point, or if for any reason the function has no limit, we say that the function is *discontinuous* at that point. The condition that $f(x)$ shall be continuous for $x = a$ is then given by the equation

$$\underset{x \doteq a}{L} f(x) = f(a).$$

This means that as x takes values arbitrarily near and if possible on both sides of the limiting value a, the corresponding values of the function $f(x)$ differ but little from $f(a)$. Whenever we are concerned merely with the existence of the limit, it is not necessary that the function be defined for $x = a$, since, as we have seen, the limit does not depend upon the value of the function at the limiting point. When the question of continuity is involved, this is no longer true, and the function must be defined at the limiting point.

As we shall see, continuity is a very important property of the functions to be discussed later. In fact, we shall confine ourselves for the most part to functions having this property. The following examples will serve to make clear the distinction between continuous and discontinuous functions.

Ex. 1. The function

$$s = f(t) = \tfrac{1}{2} gt^2$$

is continuous for all values of t; for, we have

$$\underset{t \doteq t_0}{L} \tfrac{1}{2} gt^2 = \tfrac{1}{2} gt_0^2 = f(t_0),$$

where t_0 is any definite value of t.

Ex. 2. Consider the function

$$f(x) = \frac{1}{x},$$

for values in the neighborhood of the origin. We have

$$\underset{x \doteq 0}{L} \frac{1}{x} = \infty;$$

which is another way of saying that as $x \doteq 0$, the value of the function increases without limit, and hence has no limit. The function is therefore discontinuous for $x = 0$.

We have a similar condition in Boyle's law which is expressed by the equation $pv = k$ or $p = k\,\dfrac{1}{v}$. If the volume v be diminished indefinitely, the pressure increases without limit and hence has a discontinuity for $v = 0$.

Ex. 3. Suppose a given mass of ice at a temperature below 32° F. to be heated. As heat is applied, the temperature of the ice rises, and the quantity of heat Q is a function of the temperature τ. So long as τ remains below the melting-point the function is continuous. But when the melting-point 32° F. is reached, a quantity of heat represented by AB, Fig. 9, is absorbed without any change in temperature. Likewise, when the water reaches the boiling-point, heat represented by CD is absorbed without change of temperature. At these particular temperatures the function is therefore discontinuous.

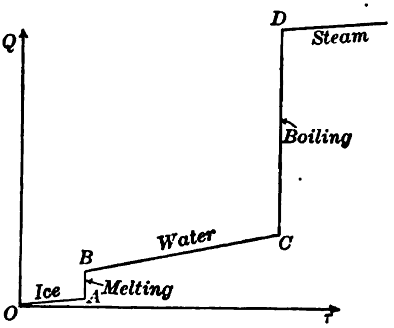

Fig. 9.

Two things are essential to the continuity of a function, viz., that the function shall have a limit, and that the value of the limit shall be the same as the value of the function at the limiting point. If either of these conditions is not fulfilled, we have a discontinuity. In the second example, the discontinuity occurred because the function had no limit for $x = 0$. A function always has a discontinuity for those values of the variable for which it becomes infinite. Again, a function may have a limit which differs from the value of the function at the critical point, thus rendering the function discontinuous at that point.

As we have seen (Art. 8), the existence of a limit requires that the limit of the values of the function for values of the variable on the one side of the critical value must equal the limit of the function for values on the other side. If this is not the case, a discontinuity exists. In other words, continuity requires that the same limiting value of the function be obtained by allowing the variable to approach its limit in all possible ways. This is illustrated by the third example.

In the preceding discussion, we have spoken only of the continuity of a function at a single point. If a function is continuous at all points of an interval, then it is said to be *continuous throughout the interval.*

In plotting a graph, it is of great assistance to know that the graph represents a continuous function and has a definite direction at each point; for then we need only to locate points of the graph sufficiently dense to give the general outline and to draw through these points a continuous curve. Since it is obviously impossible to locate all of the points of a curve, this is, in fact, the only way in which a graph can be drawn. In case the function has discontinuities, care must be taken to locate points sufficiently dense in the neighborhood of such points to determine the character of the discontinuity.*

EXERCISES.

1. Which of the trigonometric functions have points of discontinuity?

2. Find the values of x for which the function $\frac{7x+9}{x^2-5x+6}$ is discontinuous.

3. If $f(x)$ and $\phi(x)$ are integral rational functions, under what conditions is the function $\frac{f(x)}{\phi(x)}$ discontinuous?

4. Examine for continuity the functions

$$\text{(a) } y = \text{arc tan}\,\frac{1}{x}; \qquad \text{(b) } y = \frac{1}{1+e^{\frac{1}{x}}}.$$

5. From any text-book on physics obtain the proper data and draw a curve showing the change of volume of water with the temperature. Start with ice and end with steam. Show the points of discontinuity.

11. Laws of operation with limits. In this section, we shall consider several properties of limits which will be made use of in later discussions. These are given in the following theorems.

THEOREM I. *The limit of a constant times a function is equal to the constant times the limit of the function; that is,*

$$\underset{x \doteq a}{L}\, c \cdot f(x) = c \cdot \underset{x \doteq a}{L}\, f(x).$$

Let

$$\underset{x \doteq a}{L}\, f(x) = A. \tag{1}$$

We may then write

$$f(x) = A + \epsilon(x), \tag{2}$$

where $\epsilon(x)$ can be made as small as we please by taking x sufficiently near to a. We have then

$$c \cdot f(x) = c \cdot A + c \cdot \epsilon(x). \tag{3}$$

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* Not all continuous functions can be represented by smooth curves; it is possible to have a function which is continuous throughout a given interval, but which oscillates in such a manner as not to have a definite direction at any point of the interval. Such functions may be found in any treatise on the theory of functions.

Since we can make $\epsilon(x)$ as small as we please by taking x sufficiently near to a, we can also make $c \cdot \epsilon(x)$ arbitrarily small in the same way. We have then from the definition of a limit

$$\underset{x \doteq a}{L}\, c \cdot f(x) = c \cdot A = c \cdot \underset{x \doteq a}{L}\, f(x), \tag{a}$$

as the theorem requires.

THEOREM II. *If each of two functions approaches a limit, the limit of the sum (or difference) of the functions is equal to the sum (or difference) of the limits.*

Let $$\underset{x \doteq a}{L}\, f(x) = A, \qquad \underset{x \doteq a}{L}\, \phi(x) = B. \tag{1}$$

We may then write

$$f(x) = A + \epsilon_1(x) \tag{2}$$
$$\phi(x) = B + \epsilon_2(x), \tag{3}$$

where ϵ_1, ϵ_2 are arbitrarily small for values of x sufficiently near a. Combining these equations, we have

$$f(x) \pm \phi(x) = A \pm B + [\epsilon_1(x) \pm \epsilon_2(x)]. \tag{4}$$

Since, however, $\epsilon_1(x) \pm \epsilon_2(x)$ can be made as small as we choose, we have from the definition of a limit

$$\underset{x \doteq a}{L}\, [f(x) \pm \phi(x)] = A \pm B = \underset{x \doteq a}{L}\, f(x) \pm \underset{x \doteq a}{L}\, \phi(x). \tag{b}$$

It is to be noted that the converse proposition does not hold; that is, the sum or difference of two functions may have a limit, and yet one or both of the separate functions may not have a limit. In such a case the sum of the limits has no meaning.

Theorem II may be extended to cover the sum of any finite number of functions. However, it may not hold for the sum of an infinite number of functions, as the following example shows:

Given $f(n) = \frac{a}{n} + \frac{a}{n} + \frac{a}{n} + \ldots$ to n terms.

We have then

$$\underset{n \doteq \infty}{L} \left(\frac{a}{n} + \frac{a}{n} + \frac{a}{n} + \ldots \text{ to } n \text{ terms} \right) = \underset{n \doteq \infty}{L}\, \frac{na}{n} = a.$$

On the other hand, the sum of the limits gives

$$\underset{n \doteq \infty}{L}\, \frac{a}{n} + \underset{n \doteq \infty}{L}\, \frac{a}{n} + \ldots = 0 + 0 + \ldots = 0;$$

that is, the limit of the sum differs from the sum of the limits.

THEOREM III. *If each of two functions has a limit, the limit of the product of the functions is equal to the product of the limits.*

Given $$\underset{x \doteq a}{L} f(x) = A, \qquad \underset{x \doteq a}{L} \phi(x) = B. \tag{1}$$

From these limits we have at once the equations

$$f(x) = A + \epsilon_1(x) \tag{2}$$
$$\phi(x) = B + \epsilon_2(x), \tag{3}$$

where as before ϵ_1, ϵ_2 may be made arbitrarily small by confining the values of x to a sufficiently small region in the vicinity of a. We have by multiplication

$$f(x) \cdot \phi(x) = AB + [B \cdot \epsilon_1(x) + A \cdot \epsilon_2(x) + \epsilon_1(x) \cdot \epsilon_2(x)]. \tag{4}$$

The expression contained in the brackets can be made as small as we please by making ϵ_1 and ϵ_2 sufficiently small. Hence, from the definition of a limit, we have

$$\begin{aligned} \underset{x \doteq a}{L} f(x) \cdot \phi(x) &= AB \\ &= \underset{x \doteq a}{L} f(x) \cdot \underset{x \doteq a}{L} \phi(x). \end{aligned} \tag{c}$$

The converse of this theorem does not hold. There are cases where the product of two functions has a limit, but the product of the limits has no significance, because the separate functions involved have no limits. The theorem may be extended to the product of any finite number of functions.

When Theorem III is extended to the product of a finite number n of *equal* functions, we have the following:

Corollary. If a function has a limit, the limit of the nth power of the function is equal to the nth power of its limit.

THEOREM IV. *If each of two functions has a limit, the limit of the quotient of the functions is equal to the quotient of the limits, provided the limit of the denominator is different from zero.*

Given $\underset{x \doteq a}{L} f(x) = A, \qquad \underset{x \doteq a}{L} \phi(x) = B;$ we are to show that

$$\underset{x \doteq a}{L} \frac{f(x)}{\phi(x)} = \frac{\underset{x \doteq a}{L} f(x)}{\underset{x \doteq a}{L} \phi(x)},$$

where $B \neq 0$.

From the given limits, we have

$$f(x) = A + \epsilon_1(x) \tag{1}$$
$$\phi(x) = B + \epsilon_2(x), \tag{2}$$

$\epsilon_1(x)$, $\epsilon_2(x)$ being arbitrarily small. By division, we have

$$\frac{f(x)}{\phi(x)} = \frac{A + \epsilon_1(x)}{B + \epsilon_2(x)} = \frac{A}{B} + \frac{A + \epsilon_1(x)}{B + \epsilon_2(x)} - \frac{A}{B} = \frac{A}{B} + \frac{B\epsilon_1(x) - A\epsilon_2(x)}{B\,(B + \epsilon_2(x))}. \qquad (3)$$

By the proper selection of x, the denominator of the last fraction of (3) can be made as near B^2 as we please, and the numerator as small as we please. Hence the fraction can be made as small as we please, and we may write

$$\underset{x \doteq a}{L}\,\frac{f(x)}{\phi(x)} = \frac{A}{B} = \frac{\underset{x \doteq a}{L}\,f(x)}{\underset{x \doteq a}{L}\,\phi(x)}. \qquad (d)$$

The converse of the above theorem does not hold; that is, the quotient $\dfrac{f(x)}{\phi(x)}$ may have a limit without the separate functions having limits. In this case the quotient of the limits has no significance.

The question arises as to whether the limit of a function is equal to the same function of the limit; for example, whether

$$\underset{x \doteq a}{L}\sin(x^2 + 3) = \sin\underset{x \doteq a}{L}(x^2 + 3), \quad \text{or } \underset{x \doteq a}{L}\log x^3 = \log\underset{x \doteq a}{L}x^3.$$

By making use of the definition of continuity given in the previous article, we can answer this question by the following theorem.

THEOREM V. *If* $F(y)$ *is a continuous function of* y, *and* y *is any function of* x, *say* $y = \phi(x)$, *such that* $\underset{x \doteq a}{L}\,\phi(x) = b$,

then

$$\underset{x \doteq a}{L}\,F[\phi(x)] = F[\underset{x \doteq a}{L}\,\phi(x)].$$

We have by hypothesis,

$$\underset{x \doteq a}{L}\,y = \underset{x \doteq a}{L}\,\phi(x) = b, \qquad (1)$$

and by reason of the continuity of $F(y)$ we have also

$$\underset{y \doteq b}{L}\,F(y) = F(b). \qquad (2)$$

We may therefore make $F(y)$ differ by as little as we please from $F(b)$ by properly restricting y to values in the neighborhood of b. From (1), however, $y \doteq b$ as $x \doteq a$; hence for all values of x sufficiently close to a, $F(y) = F[\phi(x)]$ differs by as little as we please from $F(b)$. We may write therefore

$$\underset{x \doteq a}{L}\,F[\phi(x)] = F(b) = F[\underset{x \doteq a}{L}\,\phi(x)]. \qquad (e)$$

Ex. Find the limit $\underset{x \doteq 2}{L} \log (x^2 + 2x + 1)$.

Since the logarithm is a continuous function for all values different from zero, we may apply the above theorem and write

$$\underset{x \doteq 2}{L} \log (x^2 + 2x + 1) = \log \underset{x \doteq 2}{L} (x^2 + 2x + 1) = \log 9.$$

In fact, we may always write

$$\underset{x \doteq a}{L} \log \phi(x) = \log \underset{x \doteq a}{L} \phi(x),$$

provided the limit $\underset{x \doteq a}{L} \phi(x)$ is different from zero.

Theorem V may be stated in a slightly different form which is sometimes useful, namely:

THEOREM V′. *Given any continuous function of y. If for y we substitute any function of x such that $y \doteq b$ as $x \doteq a$, the limit of the resulting function as $x \doteq a$ is equal to the limit of the original function as $y \doteq b$.*

For we have

$$\underset{x \doteq a}{L} F[\phi(x)] = F[\underset{x \doteq a}{L} \phi(x)] = F(b) = \underset{y \doteq b}{L} F(y). \qquad (f)$$

The theorem holds, of course, for the special case in which y is a continuous function of x. The function resulting from the substitution is then a continuous function, as the student may easily prove.

Ex. 1. $\underset{y \doteq b}{L} \log y = \underset{x \doteq 1}{L} \log (bx) = \log b + \underset{x \doteq 1}{L} \log x.$

Ex. 2. Given $\underset{y \doteq 0}{L} \dfrac{a^y - 1}{y} \cdot$ Put $a^y - 1 = x$, whence $x \doteq 0$ as $y \doteq 0$.

Then we have

$$\underset{y \doteq 0}{L} \frac{a - 1}{y} = \underset{x \doteq 0}{L} \frac{x}{\log_e (1 + x)} \cdot$$

THEOREM VI. *If a given function lies between the values of two other functions which approach a common limit, the given function approaches the same limit.*

Suppose we have

$$\phi(x) \leqq f(x) \leqq \psi(x),$$

where

$$\underset{x \doteq a}{L} \phi(x) = \underset{x \doteq a}{L} \psi(x) = A; \qquad (1)$$

we are to show that

$$\underset{x \doteq a}{L} f(x) = A.$$

From (1) we may write

$$\phi(x) = A + \epsilon_1(x) \tag{2}$$

$$\psi(x) = A + \epsilon_2(x), \tag{3}$$

where ϵ_1 and ϵ_2 can be made as small as we please by taking x sufficiently near to a. Hence for all values of x in the neighborhood of a, we have

$$A + \epsilon_1(x) \leqq f(x) \leqq A + \epsilon_2(x),$$

whence

$$\epsilon_1(x) \leqq f(x) - A \leqq \epsilon_2(x). \tag{4}$$

But as ϵ_1, ϵ_2 can be made arbitrarily small, it follows from (4) and (3) that $f(x) - A$ can be made to differ in absolute value as little as we please from zero by the proper selection of the value of x; that is,

$$|f(x) - A| < \sigma,$$

where σ is an arbitrarily small positive number. By the definition of a limit, we then have

$$\underset{x \doteq a}{L} f(x) = A = \underset{x \doteq a}{L} \phi(x). \tag{g}$$

Ex. Show that $\underset{\theta \doteq 0}{L} \dfrac{\sin\theta}{\theta} = 1$, and $\underset{\theta \doteq 0}{L} \dfrac{\tan\theta}{\theta} = 1$.

In Fig. 10, an arc AD is described with a radius $OA = r$, and angle $AOD = \theta$. Then we have

$$AB = r\sin\theta,$$

$$\text{arc } AD = r\theta,$$

$$AC = r\tan\theta.$$

From geometry *

$$AB < AD < AC,$$

or

$$r\sin\theta < r\theta < r\tan\theta.$$

Dividing by $r\sin\theta$, we get

$$1 < \frac{\theta}{\sin\theta} < \sec\theta.$$

Now $\underset{\theta \doteq 0}{L} \sec\theta = 1$; hence since the fraction $\dfrac{\theta}{\sin\theta}$ lies between two values whose limits are equal, its limit must be the same, that is,

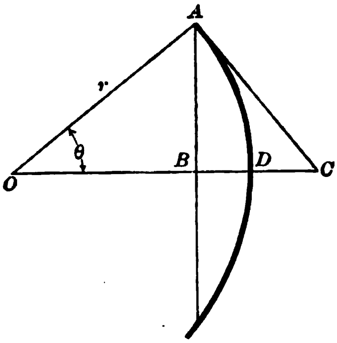

Fig. 10.

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* See Holgate's Geometry, Arts. 356–361.

$$\underset{\theta \doteq 0}{L} \frac{\theta}{\sin \theta} = \underset{\theta \doteq 0}{L} \frac{\sin \theta}{\theta} = 1.$$

Dividing the members of the inequality by $r \tan \theta$, we get

$$\cos \theta < \frac{\theta}{\tan \theta} < 1.$$

Since $\underset{\theta \doteq 0}{L} \cos \theta = 1$, it follows that

$$\underset{\theta \doteq 0}{L} \frac{\theta}{\tan \theta} = 1 = \underset{\theta \doteq 0}{L} \frac{\tan \theta}{\theta}.$$

12. Limit of a monotone function. It is not always easy to find a limit of a function. Moreover, in some discussions we are not so much concerned as to what the limit is as to whether the given function *has* a limit. The following propositions, which we give without proof,* will aid us in answering this question for certain types of functions.

(a) *If a function never decreases but always remains less than some constant, then it has a limit.*

(b) *If a function never increases but always remains greater than some constant, then it has a limit.*

Functions of the kind described in (a) and (b) are called *monotone functions.*

The following illustration will perhaps aid in making clear the significance of these statements.

Given

$$\begin{aligned} f(n) &= .333\ldots\ldots \\ &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots + \frac{3}{10^n} + \ldots \end{aligned}$$

As n increases $f(n)$ never decreases, but always remains less than .4. For $n = 2, f(n)$ lies between .3 and .4; for $n = 3$, between .33 and .34; for $n = 4$, between .333 and .334, etc. It is evident that as n increases, the interval within which $f(n)$ must lie constantly decreases. Consequently, there must be some constant such that we can make $f(n)$ differ from it by as little as we please by giving n an integral value sufficiently large. We know in this case that the constant in question is $\frac{1}{3}$, but that is not essential in establishing the fact that $f(n)$ has a limit as n becomes infinite.

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* Geometrical considerations make these propositions plausible. The formal proof, while not difficult, is scarcely within the scope of this book. See Pierpont's *Theory of Functions*, Art. 108.

EXERCISES.

Verify the following:

1. $\underset{x \doteq 3}{L} \dfrac{x^3 - 3x + 7}{x^2 - 5} = \dfrac{25}{4}$.

2. $\underset{h \doteq 0}{L} \dfrac{(x - h)^2 - 3hx}{x(x + h)} = 1$.

3. $\underset{x \doteq y}{L} \sin \dfrac{x + y}{2} \cos \dfrac{x - y}{2} = \sin y$.

4. $\underset{x \doteq 0}{L} \dfrac{\log (a + x)}{e^x} = \log a$.

5. $\underset{x \doteq 4}{L} \log \dfrac{(x^2 - 6)}{x + 1} = \log 2$.

6. $\underset{\theta \doteq \frac{\pi}{2}}{L} \log \sin \theta = 0$.

7. $\underset{x \doteq 2}{L} [\log (x^2 - 1) - \log (x - 1)] = \log 3$.

8. Give graphical illustrations of theorems (a) and (b), Art. 12.

13. Indeterminate forms. It may happen that for some value of the variable the function takes an illusory or indeterminate form, as $\frac{0}{0}$. Consider, for example, the function $y = \dfrac{x^2 - 4}{x - 2}$. For every value of the variable x other than $x = 2$, the function has a definite value, but for $x = 2$ it becomes $\dfrac{4 - 4}{2 - 2} = \dfrac{0}{0}$. Strictly speaking, the function $\dfrac{x^2 - 4}{x - 2}$ has no definite value for $x = 2$; hence in order to define the function completely for every value of the variable, we must assign it a value for this particular value of the variable. To guide us in our definition, we make use of a simple graphical representation. For x different from $x = 2$, the equation

$$y = \frac{x^2 - 4}{x - 2}$$

reduces to

$$y = x + 2, \qquad (1)$$

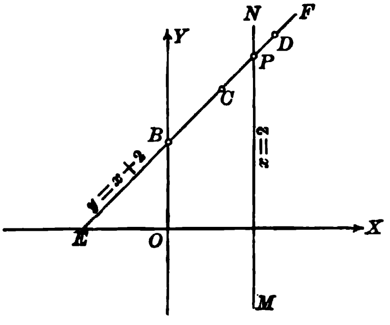

Fig. 11.

whose graph is the line EF, Fig. 11. Hence for values of x other than 2, the values of the function are represented graphically by points on the line EF, as B, C, D. But when $x = 2$, the function has no definite value, and hence we may represent it by any point whatever on the line $x = 2$, that is, on MN. Of the values which might be assigned to the function for

$x = 2$, there is one value, represented by the point P, which is the limit of the values represented by the points on EF as x approaches 2; and this value of y is defined as the value of the function when $x = 2$.

In other words, we define the value of the function for the critical value of the variable as the limit which the function approaches as the variable approaches the value for which the function becomes indeterminate. Thus for $x = 2$, the value of the function $\frac{x^2 - 4}{x - 2}$ is defined as $\underset{x \doteq 2}{L} \frac{x^2 - 4}{x - 2}$.

In general, if $f(x)$ is indeterminate for any value $x = a$, the value of the function for $x = a$ is defined by the limit

$$\underset{x \doteq a}{L} f(x).$$

The complete definition of a function when it assumes an indeterminate form involves, therefore, the determination of a limit. Any definition of a function for those values of the variable for which it becomes indeterminate is of course merely a convention; but in this case it is a very useful convention, because by it the function becomes continuous for those values.

Among the indeterminate forms that a function may assume are the following:

$$\frac{0}{0}; \quad \frac{\infty}{\infty}; \quad \infty - \infty; \quad 0 \times \infty; \quad 0^0; \quad \infty^0; \quad 1^\infty.$$

An example of the first form is given by the function $\frac{\sin x}{x}$ which for $x = 0$ takes the form $\frac{0}{0}$. The value of this function for $x = 0$ is therefore

$$\underset{x \doteq 0}{L} \frac{\sin x}{x} = 1. \qquad \text{(Art. 11)}$$

The function $x \log x$ for $x = 0$ has the form $0 \times \infty$, and the function $(1 + x)^{\frac{1}{x}}$ for $x = 0$ has the form 1^∞. By proper transformations all these indeterminate forms may be reduced to the form $\frac{0}{0}$.

In many cases the limits are easily found by obvious algebraic transformations or by the use of series. In the evaluation of the indeterminate form of a rational function, that is, the quotient of two polynomials, the following principles may be used to advantage.

1. If the function is of the form $\frac{\phi(x)}{\psi(x)}$ and becomes $\frac{0}{0}$ for $x = 0$, divide both numerator and denominator by the lowest power of x that occurs in either numerator or denominator. If the fraction becomes $\frac{\infty}{\infty}$ for $x = \infty$, divide both terms of the fraction by the highest power of x in either numerator or denominator.

2. If the function has the form $\frac{\phi(x)}{\psi(x)}$ and takes the form $\frac{0}{0}$ for $x = a$, divide both $\phi(x)$ and $\psi(x)$ by the highest power of $(x - a)$, common to both.

The student should note that we do not divide the terms of the fraction by zero or infinity. For example, while the division by the factor $(x - a)$ holds only for x different from a, the values of x may, however, be taken as close as we please to a. Hence, dividing first by this factor and afterwards passing to the limit, we obtain the proper result. Indeterminate forms will be considered again in Chapter XV.

The following examples illustrate some methods that may be used in evaluating various indeterminate forms.

Ex. 1. Find the value of $\frac{x^2 - 6x}{x^3 - 4x^2 - 12x}$ for $x = 0$.

$$\underset{x \doteq 0}{L} \frac{x^2 - 6x}{x^3 - 4x^2 - 12x} = \underset{x \doteq 0}{L} \frac{x - 6}{x^2 - 4x - 12} = \frac{1}{2}.$$

Ex. 2. Evaluate the function $\frac{2x^2 + 3x^3}{x + 5x^3}$ for $x = \infty$.

$$\underset{x = \infty}{L} \frac{2x^2 + 3x^3}{x + 5x^3} = \underset{x = \infty}{L} \frac{\frac{2}{x} + 3}{\frac{1}{x^2} + 5} = \frac{0 + 3}{0 + 5} = \frac{3}{5}.$$

Ex. 3. Evaluate $\frac{x^2 - 4}{x - 2}$ for $x = 2$.

$$\underset{x \doteq 2}{L} \frac{x^2 - 4}{x - 2} = \underset{x \doteq 2}{L} (x + 2) = 4.$$

Ex. 4. Find the value of $\frac{x^3 - 3x^2 + 5x - 3}{2x^3 - 5x^2 + 8x - 5}$ for $x = 1$.

$$\frac{x^3 - 3x^2 + 5x - 3}{2x^3 - 5x^2 + 8x - 5} = \frac{(x^2 - 2x + 3)(x - 1)}{(2x^2 - 3x + 5)(x - 1)} = \frac{x^2 - 2x + 3}{2x^2 - 3x + 5}.$$

Hence $$\underset{x \doteq 1}{L} \frac{x^2 - 2x + 3}{2x^2 - 3x + 5} = \frac{1}{2}.$$

Ex. 5. Evaluate $\dfrac{a - \sqrt{a^2 - x^2}}{x^2}$ for $x = 0$.

This function takes the form $\dfrac{0}{0}$. To find its value for $x = 0$, we multiply both terms of the fraction by the complementary surd $a + \sqrt{a^2 - x^2}$. We then have

$$\frac{a - \sqrt{a^2 - x^2}}{x^2} = \frac{a^2 - (a^2 - x^2)}{x^2\,(a + \sqrt{a^2 - x^2})} = \frac{x^2}{x^2\,(a + \sqrt{a^2 - x^2})} = \frac{1}{a + \sqrt{a^2 - x^2}},$$

and

$$\underset{x \doteq 0}{L}\, \frac{a - \sqrt{a^2 - x^2}}{x^2} = \underset{x \doteq 0}{L}\, \frac{1}{a + \sqrt{a^2 - x^2}} = \frac{1}{2a}.$$

Ex. 6. Evaluate the function $\dfrac{1 - \cos\theta}{\theta^2}$ for $\theta = 0$.

Since $1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}$, we have

$$\frac{1 - \cos\theta}{\theta^2} = \frac{2\sin^2\dfrac{\theta}{2}}{\theta^2} = \frac{1}{2}\,\frac{\sin^2\dfrac{\theta}{2}}{\left(\dfrac{\theta}{2}\right)^2} = \frac{1}{2}\left[\frac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}}\right]^2.$$

But

$$\underset{\theta \doteq 0}{L}\, \frac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}} = 1; \qquad \text{(Art. 11.)}$$

hence we obtain

$$\underset{\theta \doteq 0}{L}\, \frac{1 - \cos\theta}{\theta^2} = \frac{1}{2}\,\underset{\theta \doteq 0}{L}\left[\frac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}}\right]^2 = \frac{1}{2}.$$

Ex. 7. Evaluate $\dfrac{1 - \cos\theta}{\theta}$ for $\theta = 0$.

As in Ex. 6,

$$\frac{1 - \cos\theta}{\theta} = \frac{2\sin^2\dfrac{\theta}{2}}{\theta} = \frac{\theta}{2}\left[\frac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}}\right]^2.$$

$$\underset{\theta \doteq 0}{L}\, \frac{\theta}{2}\left[\frac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}}\right]^2 = \underset{\theta \doteq 0}{L}\, \frac{\theta}{2}\,\underset{\theta \doteq 0}{L}\left[\frac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}}\right]^2 = 0.$$

Hence for $\theta = 0$, $\dfrac{1 - \cos\theta}{\theta} = 0$.

Ex. 8. Evaluate the function $\sec x - \tan x$ for $x = \dfrac{\pi}{2}$.

The form taken in this case is $\infty - \infty$. By trigonometric transformation we have

$$\sec x - \tan x = \frac{(\sec x - \tan x)(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \frac{\sec^2 x - \tan^2 x}{\sec x + \tan x} = \frac{1}{\sec x + \tan x},$$

and $$\underset{x \doteq \frac{\pi}{2}}{L} \frac{1}{\sec x + \tan x} = L \frac{1}{\infty} = 0.$$

EXERCISES.

Find the limiting values of the following functions for the given values of the variables.

1. $\dfrac{6x^3 + 3x}{2x^3 + 4x^2 - 5x}$, $x = 0$.

2. $\dfrac{x^3 - 3x^2 - 4x + 12}{x^2 + 4x - 21}$, $x = 3$.

3. $\sqrt{1 + x} - \sqrt{x}$, $x = \infty$.

4. $\dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{x}$, $x = 0$.

5. $\dfrac{x^3 - 64}{x - 4}$, $x = 4$.

6. $\dfrac{x^7 - a^7}{x - a}$, $x = a$.

7. Show that $\underset{\theta \doteq 0}{L} \dfrac{\sin n\theta}{\sin \theta} = n$.

Suggestion: $$\frac{\sin n\theta}{\sin \theta} = n \frac{\frac{\sin n\theta}{n\theta}}{\frac{\sin \theta}{\theta}}, \quad L \frac{\frac{\sin n\theta}{n\theta}}{\frac{\sin \theta}{\theta}} = \frac{L \frac{\sin n\theta}{n\theta}}{L \frac{\sin \theta}{\theta}} = \frac{1}{1}.$$

8. Show that for $\theta = 0$, $\dfrac{1 - \cos \theta}{\sin \theta} = 0$.

9. Show that $\underset{\theta \doteq \frac{\pi}{2}}{L} \dfrac{\sec \theta}{\tan \theta} = 1$.

10. Show that $\underset{x \doteq 0}{L} \dfrac{1}{x} \sin \dfrac{x}{2} = \dfrac{1}{2}$.

11. Evaluate $\dfrac{x^3 + 3x^2 - 5x}{3x^4 - 2x^3 + 6x}$ for $x = 0$ and $x = \infty$.

12. Evaluate $\underset{x \doteq 0}{L} \dfrac{\text{arc} \tan x}{x}$.

13. Show that $\underset{\theta \doteq 0}{L} \dfrac{1 - \cos \theta}{\sin^2 \theta} = \dfrac{1}{2}$.

14. Special limits.* Certain limits of fundamental importance in the calculus demand special discussion. Among these are the following:

(a) $\underset{x \doteq a}{L} \dfrac{x^n - a^n}{x - a}$, where n is any positive integer.

By division, we have

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + \ldots + x a^{n-2} + a^{n-1}.$$

$$\underset{x \doteq a}{L} \frac{x^n - a^n}{x - a} = \underset{x \doteq a}{L} (x^{n-1} + x^{n-2}a + \ldots + x a^{n-2} + a^{n-1}).$$

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* If thought best, the exercises of this article may be taken later in connection with their applications.

In the parenthesis there are n terms, and the limit of each as $x \doteq a$ is a^{n-1}. Since n is finite, the limit of the sum is equal to the sum of the limits, and we have

$$\underset{x \doteq a}{L} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

This limit holds equally well when n is negative or fractional, and consequently for any rational number.*

(b) $\underset{h \doteq 0}{L} \frac{(x + h)^n - x^n}{h}$, n being a rational number.

From the preceding proof,

$$\underset{y \doteq x}{L} \frac{y^n - x^n}{y - x} = nx^{n-1}.$$

Let $y = x + h$; then as $y \doteq x, h \doteq 0$. Substituting, we get

$$\underset{h \doteq 0}{L} \frac{(x + h)^n - x^n}{h} = nx^{n-1}. \quad \text{(See Th. V}', \text{Art. 11.)}$$

(c) $\underset{x = \infty}{L} \left(1 + \frac{1}{x}\right)^x$.

This limit demands special attention. Let us take two numbers a and b such that

$$a > b > 0 \tag{1}$$

and let n be any positive whole number. Then

$$\frac{a^{n+1} - b^{n+1}}{a - b} = a^n + a^{n-1} b + a^{n-2} b^2 + \ldots + b^n.$$

Since $a > b$, each term of the second member after the first is smaller than a^n, and therefore the sum of the $n + 1$ terms of the second member is smaller than $a^n(n + 1)$. Hence, we have

$$\frac{a^{n+1} - b^{n+1}}{a - b} < a^n(n + 1),$$

or
$$a^{n+1} - b^{n+1} < a^n(a - b)(n + 1).$$

By transposition this inequality is readily reduced to

$$a^n [a - (a - b)(n + 1)] < b^{n+1}, \tag{2}$$

which holds for any numbers a and b that satisfy (1) and for any integral value of n. Let us now choose

$$a = 1 + \frac{1}{n}, \quad b = 1 + \frac{1}{n + 1}.$$

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* See Chrystal's Algebra, Chap. XXV, Art. 12.

These satisfy the required conditions, and when substituted in (2) give as the resulting inequality

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}. \tag{3}$$

This inequality shows that as n is increased, the value of the function $\left(1 + \frac{1}{n}\right)^n$ increases. Thus, for the smaller values of n, we have

$n =$	1	2	3	4
$\left(1 + \frac{1}{n}\right)^n =$	2	2.25	2.37 . . .	2.44 . . .

Let us now choose for a and b

$$a = 1 + \frac{1}{2n}, b = 1.$$

Substituting these in (2), we have

$$\frac{1}{2}\left(1 + \frac{1}{2n}\right)^n < 1,$$

and squaring both members, we obtain

$$\left(1 + \frac{1}{2n}\right)^{2n} < 4,$$

whence it follows that

$$\left(1 + \frac{1}{n}\right)^n < 4.$$

We have shown that the values assumed by the function $\left(1 + \frac{1}{n}\right)^n$ form an increasing sequence of numbers all of which lie between 2 and 4. As n is indefinitely increased, therefore, the function has by Art. 12 (a), a definite limit lying between 2 and 4. The symbol e is used to denote this limit. Hence

$$\underset{n=\infty}{L}\left(1 + \frac{1}{n}\right)^n = e,$$

where

$$4 > e > 2.$$

The number e can be obtained to any degree of approximation by taking n sufficiently large; thus, for $n = 100$,

$$\left(1 + \frac{1}{100}\right)^{100} = 2.704 \ldots,$$

while for $n = 1000$,

$$\left(1 + \frac{1}{1000}\right)^{1000} = 2.717 \ldots$$

A more convenient method of calculation will be given hereafter in connection with infinite series, and it will be shown that

$$e = 2.7182818285 \ldots$$

The constant e is the base of the Napierian system of logarithms, which is the system ordinarily used in mathematical discussions.

In the preceding discussion, n was given only positive integral values, fractional values being excluded. It can be shown that the limit is the same, if instead of n we take a continuous variable x. Hence, we may write, in general,

$$\underset{x = \infty}{L}\left(1 + \frac{1}{x}\right)^{x} = e.$$

(d) If we put $x = \dfrac{1}{z}$, we get

$$\underset{z \doteq 0}{L}(1 + z)^{\frac{1}{z}} = e.$$

(e) $\underset{x \doteq 0}{L}\dfrac{a^x - 1}{x}$.

Let $y = a^x - 1$. Then $x = log_a (1 + y)$, and for $x \doteq 0$, we have $y \doteq 0$. Substituting these values, we obtain

$$\underset{x \doteq 0}{L}\frac{a^x - 1}{x} = \underset{y \doteq 0}{L}\frac{y}{\log_a (1 + y)} = \underset{y \doteq 0}{L}\frac{1}{\log_a (1 + y)^{\frac{1}{y}}}$$

$$= \frac{1}{\log_a \underset{y \doteq 0}{L}(1 + y)^{\frac{1}{y}}} = \frac{1}{\log_a e}. \qquad \text{(Art. 11.)}$$

But $\dfrac{1}{\log_a e} = \log a$; hence

$$\underset{x \doteq 0}{L}\frac{a^x - 1}{x} = \log a.$$

(f) If we substitute e for a in (e), we have

$$\underset{x \doteq 0}{L}\frac{e^x - 1}{x} = \log e = 1.$$

EXERCISES.

1. Show that $\underset{x \doteq 0}{L} \dfrac{\log(1+x)}{x} = 1.$

2. Show that $\underset{x \doteq 0}{L} \dfrac{(1+x)^k - 1}{x} = k.$ (k a positive integer.)

3. Show that $\underset{x \doteq a}{L} \dfrac{\sin x - \sin a}{x - a} = \cos a.$

4. Show that $\underset{x \doteq 1}{L} [\log(x^{\frac{3}{2}} - 1) - \log(x^{\frac{1}{2}} - 1)] = \log 3.$

MISCELLANEOUS EXERCISES.

1. If $\phi(x) = a^x$, show that $[\phi(x)]^2 = \phi(2x)$.

2. If $\phi(y) = e^y + e^{-y}$, show that $\phi(3y) = [\phi(y)]^3 - 3\phi(y)$
and $\phi(x+y)\,\phi(x-y) = \phi(2x) + \phi(2y)$.

3. Write three rational algebraic functions for which $f(x) = f(-x)$; also three for which $f(x) = -f(-x)$. Deduce a general law for functions of these types.

4. For which of the trigonometric functions is $f(\theta) = f(-\theta)$?

5. If f and F are both increasing functions, that is, functions which increase as the variable increases, show that $f[F(x)]$ is also an increasing function.

6. Given $f(n) = a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \ldots$ to n factors, show that the limit of this product as $n = \infty$ is different from the product of the limits of the separate factors.

7. Examine the function $e^{\frac{1}{x}}$ for continuity. Draw the graph of the function from $x = -2$ to $x = +2$.

8. Find the limiting values of the following functions for the given value of the variable:

(a) $\dfrac{\tan\theta - \sin\theta}{\theta^3},\ \theta = 0.$ (b) $\dfrac{\sec\theta - 1}{\theta^2},\ \theta = 0.$

9. Show that the function

$$\frac{ae^x + be^{-x}}{e^x + e^{-x}}$$

must lie between a and b, where a and b are positive numbers and $a > b$.

10. Show that $\underset{x \doteq a}{L} [f(x)]^{\phi(x)} = \left[\underset{x \doteq a}{L} f(x)\right]^{\underset{x \doteq a}{L} \phi(x)}$ provided the second member has a determinate value.

Suggestion: The first member may be written $\underset{x \doteq a}{L} e^{\phi(x) \cdot \log f(x)}$; then theorem V, Art. 11, may be used.

11. Show that $\underset{x \doteq a}{L} \dfrac{x^n - a^n}{x - a} = na^{n-1}$, when n is a positive fraction; also when n is negative either integral or fractional.

Suggestion: For the first case, let $n = \dfrac{p}{q}$, and assume $x = z^q$ and $a = b^q$; in the second case, put $n = -m$.

12. Making use of the limit theorems, Art. 11, prove the following properties of continuous functions:

(a) The sum of a finite number of continuous functions is a continuous function throughout any interval in which all the functions are continuous.

(b) The product of a finite number of continuous functions is a continuous function throughout the common interval of continuity.

(c) Every integral rational algebraic function is continuous for all finite values of the variable.

13. Air flows from a vessel in which the pressure is p into the atmosphere. The weight of air flowing per second is given approximately by the equation

$$w = C\left(\frac{p_a}{p}\right)^{0.7}\sqrt{1 - \left(\frac{p_a}{p}\right)^{0.3}},$$

in which C is a constant depending upon the orifice, and p_a denotes atmospheric pressure. Draw the graph, taking p as the variable and letting $C = 1$ and $p_a = 15$. From the graph find the value of p for which w is greatest. What happens when $p < p_a$?

14. The equation

$$y = e^{-at}(A \sin bt + B \cos bt)$$

expresses, as a function of the time t, the displacement of a vibrating point from an initial position, when the motion is retarded by friction. Take $a = 0.4$, $b = 2.973$, $A = 5$, $B = 0$, and draw the graph.

Suggestion: Draw graphs of e^{-at} and of $(A \sin bt + B \cos bt)$ and multiply the ordinates for the same value of t.

CHAPTER III

DERIVATIVES OF ALGEBRAIC FUNCTIONS

15. Increments. The idea of a variable as defined in the previous chapter and used throughout the calculus is but little used in elementary mathematics. There the symbols x, y, etc., stood usually for unknown but fixed numbers whose values were to be determined. Here, as in the analytic geometry, we associate the same symbols with magnitudes which change or grow according to some law determined by the problem under discussion. This idea of growth is of prime importance in the calculus, and lies at the basis of the discussion in the present chapter.

A change in the variable is called an *increment,* and is the difference between the variable and some assigned value of it. For example, consider the curve $y = f(x)$, and let x_1 denote a given value of the variable. The increment of x is then $x - x_1$. We shall denote an increment of a variable by prefixing the Greek letter Δ to the variable. Thus Δx, Δt, etc., denote increments of x, t, etc. In Fig. 12 we have $\Delta x = x - x_1$.

Corresponding to an increment of a variable, there exists an *increment of a function* of that variable. This increment is obtained by taking the difference between the function and its value for the arbitrarily assigned value of the variable. For example, in Fig. 12 the increment of $f(x)$ corresponding to the increment $(x - x_1)$ of x is

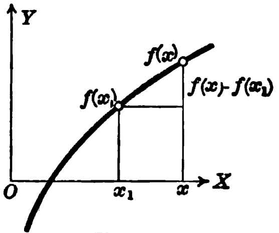

Fig. 12.

$$f(x) - f(x_1).$$

We denote the increment of the function by prefixing the letter Δ just as in the case of the increment of the variable. Thus, we have

$$\begin{aligned}\Delta f(x) &= f(x) - f(x_1)\\ &= f(x_1 + \Delta x) - f(x_1).\end{aligned}$$

If we have $y = f(x)$, then instead of writing $\Delta f(x)$ we can equally well write Δy.

Given the increment of the variable, we may calculate the corresponding increment of the function, as shown in the following examples.

Ex. 1. Given the function

$$f(x) = 3x^2 + 4x + 2.$$

Then

$$\begin{aligned} \Delta f(x) &= f(x_1 + \Delta x) - f(x_1) \\ &= 3(x_1 + \Delta x)^2 + 4(x_1 + \Delta x) + 2 - (3x_1^2 + 4x_1 + 2) \\ &= 6x_1\Delta x + 3(\Delta x)^2 + 4\Delta x. \end{aligned}$$

Ex. 2. Given the law of falling bodies $s = \frac{1}{2}gt^2$. Calculate Δs corresponding to $\Delta t = 2$, $t_1 = 1$.

We have here

$$\begin{aligned} \Delta s &= \tfrac{1}{2}g(t_1 + 2)^2 - \tfrac{1}{2}gt_1^2 \\ &= \tfrac{1}{2}g \cdot 8 = 4g. \end{aligned}$$

Increments are, in general, variables, and may be either positive or negative.

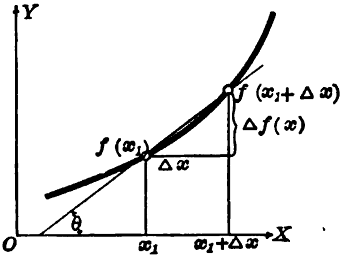

Fig. 13.

It is often desirable to compare the increment of a function with that of the variable. This comparison is most conveniently made by means of the ratio of the increments. For example, consider the curve, Fig. 13, representing the single-valued and continuous function

$$y = f(x).$$

Draw a secant line as indicated. From the figure it follows that

$$\tan\theta = \frac{\Delta f(x)}{\Delta x};$$

that is, the ratio of the increment of the function to that of the variable gives the slope of the secant line drawn through the extremities of the ordinates

$$y_1 = f(x_1), \text{ and } y = f(x_1 + \Delta x).$$

Again, suppose that we have

$$s = f(t),$$

where s is the space passed over by a moving point in the time t.

Then the ratio $\frac{\Delta s}{\Delta t}$ gives the mean or average speed of the particle for the time interval Δt.

Ex. 1. Let the equation of a parabola (Fig. 14) be $y = \frac{x^2}{20}$. Determine the slopes of secant lines drawn through a point P on the parabola whose abscissa is 10.

Let a second point Q be taken on the curve, as shown. Then $PS = \Delta x$ is the increment of the variable, and

$$SQ = \Delta y = \Delta f(x)$$

is the corresponding increment of the function. For the point P we have, by hypothesis,

$$x_1 = 10,\ y_1 = f(x_1) = \frac{10^2}{20} = 5.$$

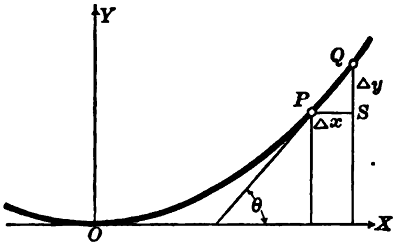

Fig. 14.

Let $\Delta x = 2$; then the abscissa of the point Q is

$$x_1 + \Delta x = 10 + 2 = 12,$$

and the ordinate is

$$\begin{aligned} y_1 + \Delta y &= f(x_1 + \Delta x) \\ &= \frac{(x_1 + \Delta x)^2}{20} \\ &= \frac{12^2}{20} = 7.2. \end{aligned}$$

Therefore
$$\begin{aligned} \Delta y &= \Delta f(x) \\ &= f(x_1 + \Delta x) - f(x_1) \\ &= 7.2 - 5 = 2.2. \end{aligned}$$

For different values of Δx the results in the following table are obtained.

Δx	$x_1 + \Delta x$	$y_1 + \Delta y$	Δy	$\tan\theta = \frac{\Delta y}{\Delta x}$
2	12	7.2	2.2	1.1
1	11	6.05	1.05	1.05
0.1	10.1	5.1005	0.1005	1.005
0.01	10.01	5.010005	0.010005	1.0005
. . .	. . . .	. . . . . .	. . . . . .	. . . . .
h	$10 + h$	$5 + h + \frac{h^2}{20}$	$h + \frac{h^2}{20}$	$1 + \frac{h}{20}$

The student will observe that, however small a value we give to Δx, the ratio $\frac{\Delta y}{\Delta x}$ has a definite value. In fact, as we allow Δx to approach zero, we have in the limit

$$\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{h \doteq 0}{L} \left(1 + \frac{h}{20}\right) = 1.$$

EXERCISES.

1. Given $f(x) = x^2 - 3x + 5$. (a) Find the increment $\Delta f(x)$ corresponding to the increment Δx of the variable. (b) Calculate $\Delta f(x)$ for $x_1 = 3$ and $\Delta x = 0.1$.

2. Given $f(x) = x^3$. (a) Find the expression for the increment $\Delta f(x)$ for $x = x_1$. (b) Find the limit of the ratio $\frac{\Delta f(x)}{\Delta x}$ as $\Delta x \doteq 0$.

3. Given $y = \sin x$. Take $x_1 = 30°$, and assume for Δx the following values: $1°$, $30'$, $5'$. Make a table showing the values of $f(x_1 + \Delta x)$, $\Delta f(x)$ or Δy, and $\frac{\Delta y}{\Delta x}$. See if the ratio approaches a limit.

4. In the equation $s = \frac{1}{2} gt^2$, take $g = 32$, $t_1 = 5$, and calculate the increment Δs corresponding to Δt. Then find the ratio $\frac{\Delta s}{\Delta t}$, and its limit as $\Delta t = 0$.

5. Find (a) the increment of volume and (b) the increment of surface when the diameter of a sphere increases from 12 to 12.05 inches.

6. Suppose that air inclosed in a cylinder has an initial volume of 10 cubic feet and an initial pressure of 20 lbs. per square inch. Assuming that air expands according to Boyle's law, viz. $pv =$ constant, calculate the value of $\frac{\Delta p}{\Delta v}$ for $\Delta v = 2, 1, 0.1, 0.01$. Determine the same ratio for $\Delta v = h$, and show that as $\Delta v \doteq 0$, this ratio has the limit -2.

16. Definition of a derivative. In the preceding article attention was called to the significance of the ratio

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x},$$

x_1 being some particular value of x. For $\Delta x = 0$ this ratio assumes the indeterminate form $\frac{0}{0}$, and to evaluate it for this value of Δx we must find its limit as $\Delta x \doteq 0$, according to the principles laid down for the evaluation of indeterminate forms. This limit is of fundamental importance in the differential calculus, and is called the *derivative* of $f(x)$ with respect to x for the value $x = x_1$. We shall denote derivatives with respect to the variables x,

t, etc., by the symbols D_x, D_t, etc. Thus we have for any value of x,

$$D_x f(x) = \underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{1}$$

The derivative may then be defined as the limit of the ratio of the increment of the function to that of the variable as the latter increment approaches the value zero.

Instead of the symbol D_x, other symbols are often employed. For example, having

$$y = f(x),$$

we may indicate the derivative by any one of the following symbols:

$$f'(x),\ y_x',\ y',\ Dy.$$

The last two are used only when there is no ambiguity as to the independent variable.

The derivative is also referred to as the *differential coefficient* or as the *derived function*,[*] and the process of finding the derivative is called *differentiation*. This process consists of the following steps:

1. Give to the independent variable an increment.
2. Calculate the corresponding increment of the function.
3. Find the ratio of the increment of the function to that of the variable.
4. Find the limit of this ratio as the increment of the variable is allowed to approach zero.

The process of differentiation does not depend, in general, upon any particular value of the independent variable and gives a general expression for the derivative applicable to all values of the independent variable. We may usually determine the value of the derivative for any particular value of the variable by substituting in the general expression for the derivative the value of the variable in question. If, however, such substitution leads to an indeterminate form, the limit

$$\underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

must be calculated for that particular value of the variable.

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[*] The word "derivative" is frequently used to mean the value of the limit (1) for a particular value of x. The term "derived function" refers more properly to the assemblage of all such values.

Ex. 1. Find the derivative of $f(x) = 5x^2 - 3x + 4$.

First, giving the variable the increment Δx, we have

$$f(x + \Delta x) = 5(x + \Delta x)^2 - 3(x + \Delta x) + 4$$
$$= 5x^2 - 3x + 4 + (10x - 3)\Delta x + 5(\Delta x)^2.$$

Subtracting, we have

$$f(x + \Delta x) - f(x) = (10x - 3)\Delta x + 5(\Delta x)^2.$$

The result of dividing by Δx is

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = 10x - 3 + 5\Delta x.$$

Taking the limit of this ratio, we obtain

$$\underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \underset{\Delta x \doteq 0}{L}(10x - 3 + 5\Delta x) = 10x - 3.$$

Hence, $f'(x) = D_x(5x^2 - 3x + 4) = 10x - 3$.

To find the value of the derivative for some particular value of x, we merely substitute that value of x in the derived function $f'(x)$. Thus, for $x = x_1$, $f'(x_1) = 10x_1 - 3$; for $x = 5$, $f'(5) = 10 \times 5 - 3 = 47$.

Ex. 2. Given the equation expressing Boyle's law, viz., $pv = k$ or $p = \frac{k}{v}$. Find the derivative of p with respect to v.

We have

$$p = \frac{k}{v},$$

whence $$p + \Delta p = \frac{k}{v + \Delta v}.$$

Subtracting, we get

$$\Delta p = \frac{k}{v + \Delta v} - \frac{k}{v} = -k\frac{\Delta v}{v(v + \Delta v)},$$

whence $$\frac{\Delta p}{\Delta v} = -k\frac{1}{v(v + \Delta v)},$$

and $$\underset{\Delta v \doteq 0}{L} \frac{\Delta p}{\Delta v} = -k \underset{\Delta v \doteq 0}{L} \frac{1}{v(v + \Delta v)} = -\frac{k}{v^2}.$$

Hence $$D_v p = f'(v) = -\frac{k}{v^2}.$$

For $k = 200$, and $v = 10$ (see Ex. 7, Art. 15)

$$f'(10) = -\frac{200}{10^2} = -2.$$

Ex. 3. Find the derivative with respect to t of the function $\frac{1}{2}gt^2$, which gives the space passed over in the time t by a falling body starting from rest.

We have

$$s = f(t) = \tfrac{1}{2}gt^2.$$

$$s + \Delta s = f(t + \Delta t) = \tfrac{1}{2}g(t + \Delta t)^2 = \tfrac{1}{2}gt^2 + gt\Delta t + \tfrac{1}{2}g(\Delta t)^2.$$

$$\Delta s = f(t + \Delta t) - f(t) = gt\Delta t + \tfrac{1}{2}g(\Delta t)^2.$$

$$\frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} = gt + \tfrac{1}{2}g\Delta t.$$

$$\underset{\Delta t \doteq 0}{L}\frac{\Delta s}{\Delta t} = \underset{\Delta t \doteq 0}{L}\frac{f(t+\Delta t)-f(t)}{\Delta t} = \underset{\Delta t \doteq 0}{L}(gt + \tfrac{1}{2}g\Delta t) = gt.$$

Hence, $$D_t s = f'(t) = gt.$$

EXERCISES.

Find the derivatives of the following functions, using the general process described in this section.

1. $y = x^5.$
2. $y = x^2 - 3x + 7$
3. $y = \dfrac{c}{x^3}.$
4. $y = \sqrt{x}.$
5. $y = 3x^2 + 9x - 1.$
6. $y = \dfrac{x - 2}{x + 2}.$
7. $y = (x - 4)^2.$
8. $y = a + bx + cx^2 + dx^3.$
9. $s = at + \frac{1}{2}gt^2$
10. $y = (x + 2)^3.$
11. $y = x^n$, where n is a positive integer.

17. Conditions for a derivative. Not every function has a derivative for all values of the variable. In order that the limit

$$\underset{\Delta x \doteq 0}{L}\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{1}$$

shall exist, it is necessary that the numerator shall approach zero simultaneously with the denominator. In other words, we must have

$$\underset{\Delta x \doteq 0}{L} f(x_1 + \Delta x) = f(x_1),$$

which is equivalent to writing

$$\underset{x \doteq x_1}{L} f(x) = f(x_1).$$

This is, however, the condition for the continuity of $f(x)$ for the value $x = x_1$. It follows that a function cannot have a derivative at a point of discontinuity.

While continuity is a necessary condition for the existence of a derivative, it must not be inferred from this that all continuous functions have derivatives. There are functions which are continuous through a given interval and yet have no derivative at any point. Such functions we shall not consider in this elementary discussion. We shall meet at most with these like $x \sin \frac{1}{x}$ which fail to have a derivative at particular points.

Although the numerator of (1) must vanish simultaneously with the denominator, it does not follow that as $\Delta x \doteq 0$, the ratio cannot become infinite, that is, cannot exceed any finite number however large. As a matter of fact, it may become infinite for particular values of the variable. Whether it does or not depends merely upon the relative rapidity with which the numerator and denominator approach their limiting value zero. We have already met with similar cases in connection with the evaluation of indeterminate forms. In such cases, we say that for the particular value of x in question, say $x = x_1$, the value of the derivative becomes infinite, and write

$$f'(x_1) = \infty.$$

Care must be taken, however, to see that as $\Delta x \doteq 0$ the ratio of the increments does not oscillate between positive and negative values which themselves increase without limit. If this is the case, we cannot properly speak of the existence of the derivative at all. For example, the ratio $\frac{1}{\Delta x} \sin \frac{1}{\Delta x}$ oscillates in the manner indicated.

18. Slope of a tangent, speed. It has been pointed out that the ratio $\frac{\Delta f(x)}{\Delta x}$ gives the slope of the secant line passing through the points whose ordinates are

$$y_1 = f(x_1), \quad y = f(x_1 + \Delta x).$$

If Δx be allowed to approach zero, the secant line approaches as a limit the tangent to the curve

$$y = f(x)$$

at the point (x_1, y_1).

Hence, if $y = f(x)$ is represented by a continuous curve, we have

$$f'(x_1) = \tan \phi,$$

where ϕ is the angle made with the axis of X by the tangent to the given curve at the point (x_1, y_1). For the sake of definiteness we shall take ϕ as the acute angle made with the positive direction of the X-axis. It may be either positive or negative. Tan ϕ is called the *slope* or *gradient* of the tangent to the curve; hence *the derivative of the function $f(x)$ for the value $x = x_1$ represents the slope of the curve $y = f(x)$ at the point (x_1, y_1).*

Again, it has been shown that $\frac{\Delta s}{\Delta t}$ gives the mean speed of the moving point during the time Δt. If Δt approaches zero, the limit, that is the value of the derivative $D_t s$ for $t = t_1$, gives the speed of the moving particle at that instant.

19. General theorems on differentiation. The following theorems, which are frequently used in the process of differentiation, are perfectly general, and consequently apply to all functions irrespective of whether they are algebraic or transcendental. Because of their fundamental importance, the student should have them at his ready command.

(a) *Derivative of a constant.*

Let $y = c.$

Then $y + \Delta y = c.$

Subtracting, $\Delta y = 0,$

whence $D_x y = \underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = 0,$

or $$\boldsymbol{D_x c = 0.} \qquad (1)$$

We have, therefore,

THEOREM I. *The derivative of a constant is zero.*

(b) *Derivative of a variable with respect to itself.*

Given $y = x,$

then $y + \Delta y = x + \Delta x,$

and $\Delta y = \Delta x.$

Hence $D_x y = \underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{\Delta x \doteq 0}{L} \frac{\Delta x}{\Delta x} = 1;$

that is $$\boldsymbol{D_x x = 1.} \qquad (2)$$

This result gives the following theorem.

THEOREM II. *The derivative of a variable with respect to itself is unity.*

(c) *Derivative of a sum.*

Let $y = f(x) + \phi(x).$

Then $y + \Delta y = f(x + \Delta x) + \phi(x + \Delta x)$

and $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}.$$

Taking the limits of both members, we have

$$\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \underset{\Delta x \doteq 0}{L} \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}.$$

or $$D_x y = D_x f + D_x \phi.$$

This process may be extended to the sum of any finite number of functions. Thus, if we have

$$y = u + v + w,$$

where u, v, w, are functions of x, we may write

$$D_x y = D_x(u + v + w) = D_x u + D_x v + D_x w. \qquad (3)$$

We may, therefore, state the following theorem.

Theorem III. *The derivative of the algebraic sum of a finite number of terms is the algebraic sum of their derivatives.*

(d) ***Derivative of a function plus a constant.***

Let $$y = f(x) + c.$$

By the application of Theorem III, we have

$$D_x y = D_x f(x) + D_x c$$
$$= D_x f(x). \qquad \text{[by (a)]}$$

That is, $$D_x[f(x) + c] = D_x f(x)$$

If u be used to denote $f(x)$, this result may be written

$$D_x(u + c) = D_x u \qquad (4)$$

We have, therefore, as a corollary to the above theorem the following:

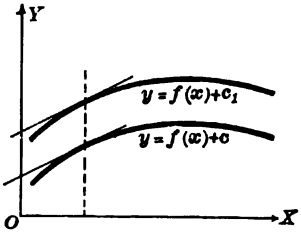

Fig. 15.

Corollary. *The derivative of a function is not affected by increasing or decreasing the function by an additive constant.*

It follows also that two functions differing only by a constant term have the same derivative. In the process of differentiating, the constant terms may consequently be neglected.

Geometrically, this corollary has an interesting significance. Suppose the function $y = f(x) + c$ to represent some curve. The effect of adding or subtracting a constant term, that is, of changing the value of c, is simply to shift the curve up or down. (See Fig. 15.) As has been shown, the derivative measures the slope of the curve, and it is evident that this slope is not changed by shifting the curve as indicated.

This corollary has also an important significance in the inverse operation of integration, as we shall see later.

(e) *Derivative of a product.*

Given $y = f(x) \cdot \phi(x)$.

We have then

$$y + \Delta y = f(x + \Delta x) \cdot \phi(x + \Delta x)$$

and
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) \cdot \phi(x + \Delta x) - f(x) \cdot \phi(x)}{\Delta x}.$$

By adding and subtracting $f(x) \cdot \phi(x + \Delta x)$ in the numerator, this may be written

$$\frac{\Delta y}{\Delta x} = \phi(x + \Delta x) \cdot \frac{f(x + \Delta x) - f(x)}{\Delta x} + f(x) \cdot \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}.$$

Since $\phi(x)$ is by hypothesis continuous, $\phi(x + \Delta x)$ in the limit becomes $\phi(x)$; hence, we have upon passing to the limit,

$$D_x y = \phi \cdot D_x f + f \cdot D_x \phi.$$

This result can be extended to the product of a finite number of functions; thus, if

$$y = uvw,$$

where u, v, w are functions of x, we have

$$D_x y = D_x(uvw) = uv \,.\, D_x w + uw \,.\, D_x v + vw \,.\, D_x u. \quad (5)$$

We have then the following theorem.

THEOREM IV. *The derivative of the product of a finite number of factors is the sum of the products obtained by multiplying the derivative of each factor by all the other factors.*

(f) *Derivative of a constant times a function.*

If we have given

$$y = c \cdot f(x),$$

we have from Theorem IV,

$$\begin{aligned} D_x y &= c \cdot D_x f(x) + f(x) \cdot D_x c \\ &= c \cdot D_x f(x). \qquad \text{[by (a)]} \end{aligned}$$

That is, $D_x c f(x) = c \cdot D_x f(x)$.

Again using u to denote $f(x)$, this result may be written

$$D_x(cu) = cD_x u. \tag{6}$$

Hence we have the following:

COROLLARY. *The derivative of a constant times a function is the constant times the derivative of the function.*

We may now combine the results of Theorem I, the corollary to Theorem II, and the above corollary in the one general statement that *differentiation destroys constant terms but retains constant factors.*

Ex. Find the derivative of $y = x^3 + 7x + 8$.

We have

$$\begin{aligned} D_x y &= D_x(x^3) + D_x(7x) + D_x 8. && \text{by (c)} \\ D_x(x^3) &= D_x(xxx) = x^2 D_x x + x^2 D_x x + x^2 D_x x && \text{by (e)} \\ &= 3x^2. && \text{by (b)} \\ D_x 7x &= 7 D_x x && \text{by Cor.} \\ &= 7. && \text{by (b)} \\ D_x 8 &= 0. && \text{by (a)} \end{aligned}$$

Combining these results, we have

$$D_x y = 3x^2 + 7.$$

The student need not write down each step as has been done in this illustrative example, but should make himself so familiar with the principles set forth in the theorems of this article that he can write down the results at once.

EXERCISES.

Find the derivatives of the following functions.

1. $y = x^3$.
2. $y = 3x^3 - 6x^2 + 7x - 1$.
3. $y = x(x^2 - 4)(x - 1) + 6$.
4. $y = x^3 - 3x^2 + 4x$.
5. $y = x^2(x^2 - 2) + x(x^2 - 1)(x + 5)$.
6. $y = 6x^3 + 5(x + 1) + x + 19$.
7. Give a geometrical interpretation of Theorem I; of Theorem II.
8. Plot the curve $y = 4x^2 + c$, giving c the values 2, 4, 6, 8, and show that each of these curves has the same slope for $x = 2$.
9. Let the distance traversed by a moving point be given as a function of the time by the equation $s = 80t - 16t^2$. Derive a general expression for the speed $D_t s$ and find the speed at the end of 3 seconds.

(g) *Derivative of a quotient.*

Let
$$y = \frac{f(x)}{\phi(x)}.$$

Proceeding as before, we have

$$y + \Delta y = \frac{f(x + \Delta x)}{\phi(x + \Delta x)},$$

and consequently

$$\begin{aligned}\Delta y &= \frac{f(x + \Delta x)}{\phi(x + \Delta x)} - \frac{f(x)}{\phi(x)}\\ &= \frac{\phi(x) \cdot f(x + \Delta x) - f(x) \cdot \phi(x + \Delta x)}{\phi(x + \Delta x) \cdot \phi(x)}.\end{aligned}$$

Adding and subtracting $\phi(x) \cdot f(x)$ in the numerator, we have, after dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{\phi(x) \cdot \dfrac{f(x + \Delta x) - f(x)}{\Delta x} - f(x) \cdot \dfrac{\phi(x + \Delta x) - \phi(x)}{\Delta x}}{\phi\,(x + \Delta x) \cdot \phi\,(x)}.$$

Remembering that $\phi(x)$ is continuous and hence as $\Delta x \doteq 0$, $\phi(x + \Delta x)$ becomes $\phi(x)$, we have for the limiting value

$$D_x y = \frac{\phi(x) \cdot D_x f(x) - f(x) \cdot D_x \phi(x)}{[\phi(x)]^2}.$$

Denoting the two functions by u and v, this result can be more conveniently written as follows:

$$D_x\left(\frac{u}{v}\right) = \frac{v\,D_x u - u\,D_x v}{v^2}. \tag{7}$$

We may state this result in the following theorem:

THEOREM V. *The derivative of a fraction is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

If the numerator of the fraction is a constant, the above result becomes

$$D_x\left(\frac{c}{v}\right) = -\frac{c\,D_x v}{v^2}, \tag{8}$$

which is frequently convenient.

Ex. 1. Find $D_x\left(\frac{1}{x}\right)$.

We have from (8) $$D_x\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Ex. 2. Given $f(x) = \dfrac{x+2}{x^2+3x+1}$; find $f'(x)$.

From Theorem V. we have

$$f'(x) = \frac{(x^2+3x+1)\,D_x(x+2) - (x+2)\,D_x(x^2+3x+1)}{(x^2+3x+1)^2}$$

$$= \frac{(x^2+3x+1)\,1 - (x+2)(2x+3)}{(x^2+3x+1)^2} = -\frac{(x^2+4x+5)}{(x^2+3x+1)^2}.$$

EXERCISES.

Find the derivatives of the following functions:

1. $y = \dfrac{x+5}{x^3}$.
2. $y = \dfrac{x^3}{x^2-3}$.
3. $y = \dfrac{1}{x^2}$.
4. $y = \dfrac{3}{(x-4)^3}$.
5. $s = \dfrac{3t}{t^2+1}$.
6. $s = \dfrac{1}{t^3-t+1}$.
7. $\rho = \theta - \dfrac{1}{\theta^2}$.
8. $\rho = \dfrac{\theta}{\theta^2-4}$.
9. $y = \dfrac{3x^2-6x+5}{x^2-7}$.
10. $y = \dfrac{x^2-2x+5}{x^3-x+8}$.
11. $y = \dfrac{x-8}{x^2+2x-5}$.

12. Show that any rational algebraic function of x may be differentiated by the application of the preceding rules.

13. Show that if the nth power of a variable occurs as a factor in the denominator of a fraction, the $n+1$st power of the factor will occur in the denominator of the derivative of the fraction after reduction to lowest terms.

20. Differentiation of algebraic functions. The definition of a derivative given in Art. 16 and the laws for differentiation developed in Art. 19 are perfectly general and apply to all functions. Having differentiated certain fundamental functions, we need only apply these laws in order to differentiate all functions ordinarily considered in a first course in calculus.

When we have once shown how to find the derivative of u^n, where u is a function of x, and n has all real values, we may then write at once the derivative of any explicit algebraic function by applying the results obtained in the preceding article. In finding the derivative of u^n, there are four cases to be considered, viz.:

1. When n is a positive integer.
2. When n is a negative integer.
3. When n is a rational fraction.
4. When n is an incommensurable number.

Following the method used to find the derivative, we have

$$y = u^n,$$
$$y + \Delta y = (u + \Delta u)^n.$$

Consequently, we have for $\Delta u = 0$

$$\frac{\Delta y}{\Delta x} = \frac{(u + \Delta u)^n - u^n}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

Passing to the limit, we have, since u is continuous and therefore Δu approaches zero with Δx,

$$\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{\Delta u \doteq 0}{L} \frac{(u + \Delta u)^n - u^n}{\Delta u} \underset{\Delta x \doteq 0}{L} \frac{\Delta u}{\Delta x}. \tag{1}$$

We have already considered a limit similar to

$$\underset{\Delta u \doteq 0}{L} \frac{(u + \Delta u)^n - u^n}{\Delta u}$$

in Art. 14 (b), where n had positive integral values, and again in Ex. 11, p. 35, for negative and rational fractional values. In each of these cases, it was shown that the limit was nu^{n-1}. In Art. 55 it will be shown that this same limit is obtained when n is any real constant, rational or irrational. We therefore write for the value of the limit of (1), replacing y by u^n,

$$\boldsymbol{D_x(u^n) = nu^{n-1} \cdot D_x u,}$$

which holds for all real values of n.

Ex. 1. Given $u = x^n$.

In this case $D_x u = 1$, and we have at once from the formula

$$D_x(x^n) = nx^{n-1}.$$

Ex. 2. Let $y = a(3x^2 + 7)^3$; find $D_x y$.

Put $u = 3x^2 + 7.$

We have then $D_x y = a \cdot D_x u^3 = 3au^2 D_x u.$

But $D_x u = D_x(3x^2 + 7) = 6x.$

Combining these results, we obtain

$$D_x y = 18ax(3x^2 + 7)^2.$$

Ex. 3. Let $y = \sqrt{x^2 - a^2}$.

Substituting $u = x^2 - a^2,$

we have $y = u^{\frac{1}{2}}.$

Hence $D_x y = \frac{1}{2} u^{-\frac{1}{2}} D_x u = \frac{1}{2} u^{-\frac{1}{2}} 2x = \dfrac{x}{\sqrt{x^2 - a^2}}.$

EXERCISES.

Differentiate the following functions:

1. $y = x^{\frac{1}{3}}$.
2. $y = x^{\frac{4}{3}} - 2x^{\frac{1}{3}}$.
3. $y = x^{-5} + 2x^{-3} - 4x^{-1}$.
4. $y = (3x - 7)^3$.
5. $y = (2x^2 - 5x + 3)^2$.
6. $y = \sqrt{1 - x^2}$.
7. $y = \sqrt[3]{a^2 + x^2}$.
8. $y = (3x - 7)^{-\frac{1}{3}}$.
9. $y = (x^3 - 6)^{-2}$.
10. $\rho = \theta^2 + k\theta^{\frac{1}{2}} - 2$.
11. $p = \frac{C}{v^n}$.
12. $y = \frac{1}{x} - \frac{3}{x^2} - \frac{8}{x^3}$.
13. $y = \frac{1}{(x - 5)^{\frac{2}{3}}}$.
14. L $y = \sqrt{\frac{1 - x^2}{1 + x^2}}$. $y' = -\frac{2x}{\sqrt{(1-x^2)(1+x^2)^3}}$
15. L $y = (a + x)\sqrt{a - x}$. $y' = \frac{a-3x}{2\sqrt{a-x}}$
16. L $y = x^p(1 - x)^q$. $x^{p-1}(1-x)^{q-1}(p-px-qx)$

17. By differentiating $\frac{1}{x^n}$, show that $D_x x^n = nx^{n-1}$ holds for negative integral values of n.

21. Derivative of a function of a function. The derivative of u^n discussed in the last article, where u is a function of x, is a special form of the more general case which we shall now consider. Let us suppose that $y = f(u)$, and $u = \phi(x)$, where these functions have respectively the derivatives $f'(u)$ and $\phi'(x)$ for the corresponding values of the variables u and x. If we wish to find $D_x y$, we can do so by substituting the given value of u in $f(u)$, thus expressing y directly as a function of x; from this expression we can obtain the desired result by methods already explained. In most cases, however, the derivative can be obtained more easily by the following method.

Let x take an increment; then u and y will also take increments, and since u and y are continuous functions of x, these increments will approach zero as the increment of x approaches zero. We have then

$$\Delta y = f(u + \Delta u) - f(u).$$

$$\frac{\Delta y}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta u} \cdot \frac{\Delta u}{\Delta x}, \quad \Delta u \neq 0.$$

Since $\Delta u \doteq 0$ as $\Delta x \doteq 0$, we have in the limit

$$\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{\Delta u \doteq 0}{L} \frac{f(u + \Delta u) - f(u)}{\Delta u} \underset{\Delta x \doteq 0}{L} \frac{\Delta u}{\Delta x};$$

that is,

$$D_x y = D_u y \cdot D_x u. \tag{1}$$

This formula expressed in words gives us the following theorem.

THEOREM. *If $y = f(u)$ and $u = \phi(x)$, the derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x.*

This theorem merely asserts the principle that if u changes m times as rapidly as x, and y changes n times as rapidly as u, then y changes mn times as rapidly as x. For example, if a horse travels twice as fast as a man, and a train three times as fast as the horse, the train travels six times as fast as the man.

Ex. 1. Find D_xy when $y = \sqrt{u}$ and $u = 3x^2 - 4$.

We have $\quad D_uy = \frac{1}{2}u^{-\frac{1}{2}}$, and $D_xu = 6x$;

hence $\quad D_xy = \frac{1}{2}u^{-\frac{1}{2}} \cdot 6x = 3xu^{-\frac{1}{2}} = \dfrac{3x}{\sqrt{3x^2 - 4}}$.

Ex. 2. Find D_xy when $y = (a^2 - x^2)^{-\frac{3}{2}}$.

Put $\quad u = a^2 - x^2$; then $y = u^{-\frac{3}{2}}$, $D_uy = -\frac{3}{2}u^{-\frac{5}{2}}$, and $D_xu = -2x$.

Therefore $\quad D_xy = 3xu^{-\frac{5}{2}} = \dfrac{3x}{(a^2 - x^2)^{\frac{5}{2}}}$.

EXERCISES.

Differentiate the following functions.

1. $y = x^5 - 3x + 2x^{-\frac{1}{2}}$.
2. $y = \sqrt{x^2 - 3}$.
3. $y = (x^2 - x + 1)^{\frac{1}{2}}$.
4. $y = \sqrt{\dfrac{x^2 - 3}{x^3 + 1}}$.
5. $y = x^{\frac{5}{2}} - x^{\frac{3}{2}} + 4x^{\frac{1}{2}} - 2$.
6. $y = (x^2 - a^2)^{\frac{1}{2}}(x + b)^{\frac{3}{2}}$.
7. $y = \left(\dfrac{x^3}{x^3 + 3a^3}\right)^{\frac{1}{3}}$.
8. $y = \sqrt[3]{6x^2 - x^3}$.
9. $y = \sqrt{\dfrac{x^3}{2a - x}}$.
10. $y = \sqrt{(a + x)(b + x)}$.
11. $y = \dfrac{x}{a\sqrt{2ax - x^2}}$.
12. $s = \dfrac{a + bt + ct^2}{t}$.
13. $s = \dfrac{3\sqrt{t}}{2 + t^2}$. $\quad \dfrac{6 - 9t^2}{2\sqrt{t}(2 + t^2)^2}$
14. $\rho = \dfrac{\theta^2}{1 - \theta}$. $\quad \dfrac{\theta(2 - \theta)}{(1 - \theta)^2}$
15. $\rho = \dfrac{\sqrt{\theta}}{1 + \theta^2}$. $\quad 1 - 3\theta^2$
16. $y = (1 - 4x^2 + 5x^4)(1 + x^3)^2$.
17. $y = 5(x^2 - 4)^{\frac{3}{2}}(x + 5)^{\frac{1}{2}}$.
18. $y = \dfrac{\sqrt{c} + 2\sqrt{x}}{2\sqrt{x}\sqrt{c^{\frac{1}{2}} + x^{\frac{1}{2}}}}$.
19. $y = \dfrac{\sqrt{ax^2 + 2bx + c}}{\sqrt{px^2 + 2qx + r}}$.
20. $y = \dfrac{a - \sqrt{a^2 - x^2}}{x^2}$.
21. By putting $f(u) = cu$, prove the corollary to Theorem IV, Art. 19.
22. By putting $f(u) = u^n$, develop the law for differentiation of u^n given in Art. 20.

23. Write a function which has x^2 as its derivative.

24. Show that the curve for which the slope of the tangent line (that is, $\tan \phi$) is numerically equal to the abscissa has the general form $y = \frac{1}{2} x^2 + C$.

25. Find the values of x for which the derivative of $f(x) = x^3 - 9x^2 + 24x$ becomes zero. What is the geometrical interpretation? $x=2$ max., $x=4$ min.

26. At what points of the curve $y = x^3 - 12x$ is the tangent parallel to the X-axis? $x = \pm 2$

22. Derivatives of inverse functions. As we have seen, if y is given as an explicit function of x, then x expressed as a function of y is the inverse function. Usually in the process of differentiation we express y as an explicit function of x and determine $D_x y$ directly. Sometimes, however, it is convenient to express x as a function of y and find $D_x y$ by means of the inverse function. To find the relation between the two derivatives, we may proceed as follows:

Given

$$y = f(x).$$

Let the inverse function be

$$x = \phi(y).$$

By differentiation, we have from this equality by aid of Art. 21,

$$1 = D_y \phi(y) \cdot D_x y,$$

or

$$1 = D_y x \cdot D_x y.$$

Hence we have

$$D_x y = \frac{1}{D_y x}. \tag{1}$$

This formula states the principle that if y is changing n times as rapidly as x, then x is changing $\frac{1}{n}$th as rapidly as y.

Ex. Given the equation of the parabola $y^2 = 4px$; find $D_x y$. Solving for x, we have

$$x = \frac{y^2}{4p},$$

whence

$$D_y x = \frac{2y}{4p} = \frac{y}{2p}.$$

From the theorem of inverse functions, we have

$$D_x y = \frac{1}{D_y x} = \frac{2p}{y} = \frac{2p}{\sqrt{4px}} = \sqrt{\frac{p}{x}}.$$

EXERCISES

1. Find $D_x y$ when $y^2 = x - 4$.
2. Find $D_x y$ when $\frac{y-2}{y+1} = x^2$.
3. If $\rho^2 = a\theta + b$, find $D_\theta \rho$.
4. If $v^2 = 2gs$, find $D_s v$.

23. Derivative of one function with respect to another, when both are functions of a common variable.

Suppose we have given

$$y = f(t), \quad x = \phi(t),$$

and that we wish to find $D_x y$. We have, for $\Delta t \neq 0$,

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}}.$$

Because of the continuity of $f(t)$ and $\phi(t)$, $\Delta y \doteq 0$ and $\Delta x \doteq 0$ simultaneously as $\Delta t \doteq 0$. Hence by Art. 16, we have upon passing to the limit

$$D_x y = \frac{D_t y}{D_t x}. \tag{1}$$

Ex. 1. Given $y = z^2 + 1$, and $x = \sqrt{z}$; find $D_x y$.

$$D_z y = 2z,$$

and

$$D_z x = \tfrac{1}{2} z^{-\frac{1}{2}}.$$

Hence

$$D_x y = \frac{2z}{\frac{1}{2} z^{-\frac{1}{2}}} = 4z^{\frac{3}{2}} = 4x^3.$$

Ex. 2. The equations $s = \frac{1}{2}gt^2$ and $v = gt$ refer to falling bodies. Find the derivative $D_s v$.

$$D_s v = \frac{D_t v}{D_t s} = \frac{g}{gt} = \frac{1}{t} = \frac{g}{v}.$$

EXERCISES.

1. Find $D_x y$, when $y = 3t^2 - t - 10$ and $x = t + 8$. 6t − 1

2. Find $D_x y$, when $y = t^4 - 3t^2 + 7$ and $x = t^2 - 2t + 4$. Find the values of this derivative for $t = 0, 2, 5$.

3. If $\rho = \sqrt{t}$ and $\theta = t^2 - 10$, find $D_\theta \rho$.

4. If $x = at$ and $y = bt - \frac{1}{2}ct^2$, find $D_x y$ and $D_y x$.

5. Work examples 3 and 4 by eliminating t before differentiating.

6. Prove the theorem of Art. 23 by making use of the theorems of Arts. 21 and 22.

MISCELLANEOUS EXERCISES.

Differentiate the following functions with respect to the variable indicated.

1. $y = 3x^4 - 6x^3 + 7x - 1.$

2. $y = x^5(x^2 - 4)(x - 1).$

3. $s = 5t^2 - 3t + 2.$

4. $\rho = \theta^2 - 2 + \frac{1}{\theta}.$

5. $s = s_0 + v_0 t + \frac{1}{2} g t^2.$

6. $y = 3x^2\sqrt{x^2 - 5}.$

7. $y = x^2(a - x)^3(b + x)^2(c - x).$

8. $y = (1 + cx^3)^{\frac{1}{n}}.$

9. $y = a + \frac{b}{x} + \frac{c}{x^2}.$

10. $y = \frac{ax + b}{cx + d}.$

11. $y = \frac{1}{x^n + c^n}.$

12. $y = (ax^2 + bx + c)^{\frac{1}{2}}.$

13. $y = \sqrt{5x^2 - 6x + 12}.$

14. $y = \frac{x^n}{(1 + x)^n}.$

15. $y = \frac{1}{x^n(a + x)^m}.$

16. $y = \frac{(x - a)(x - b)}{(x - c)(x - d)}.$

17. $y = (x + \sqrt{x^2 - 1})^n.$

18. $y = \sqrt{x + \sqrt{1 + x^2}}.$

19. $y = \sqrt{\frac{x - 1}{x + 1}}.$

20. $y = \sqrt{\frac{x^2 + 5}{x - 3}}.$

21. $y = \frac{(1 - x)\sqrt{1 - x^2}}{1 + x}.$

22. $y = \frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}.$

23. $y = \frac{x^3(x^2 - a^2)^{\frac{1}{2}}}{x + a}.$

24. $y = \frac{x}{x + \sqrt{x^2 - 1}}.$

25. $y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}.$

26. $\rho = \sqrt{a\theta} - \sqrt{b\theta^3}.$

27. $R = a(1 - c\theta + d\theta^2)^{-1}.$

28. $u = \sqrt[3]{v}\sqrt{\sqrt{v} + 1}.$

29. $y = \frac{\sqrt{x^2 + 1} - x}{\sqrt{x^2 + 1} + x}.$

30. $p = \frac{c}{v - b} - \frac{a}{v^2}.$

31. Show that the slope of the tangent to the curve $y = x^3 + 4$ is never negative. Find the slope for $x = 0$, $x = 2$. For what values of x does the slope decrease as x increases?

32. Find the angle which the tangent to the parabola $y^2 = 9x$ at the point (4, 6) makes with the X-axis.

33. Given $y = \frac{u + 5}{u^2 - 15}$ and $u = x^2 - 3$; find $D_x y$.

34. Given $s = bt + \frac{1}{2}at^2$ and $v = at$; find $D_s v$ in terms of v.

35. The equation $pv = C$ expresses Boyle's law, C being a constant. Find $D_v p$ and $D_p v$.

36. Clausius' equation, which gives the relation between the pressure, volume, and temperature of certain substances, is

$$p = \frac{RT}{v - a} - \frac{c}{T(v + b)^2},$$

in which R, a, b, and c are constants.

(a) Consider T also constant and find $D_v p$.

(b) Consider v constant and T variable and find $D_T p$.

37. From Regnault's experiments, the heat q required to raise the temperature of a unit weight of water from 0° C. to a temperature τ is given by the equation

$$q = \tau + 0.00002\,\tau^2 + 0.0000003\,\tau^3.$$

Find $D_\tau q$ and calculate its numerical value for $\tau = 35°$.

38. In expanding, a gas follows the law $pv^n = C$, in which n and C are constants. Find $D_v p$ and $D_p v$.

39. Find the slope of the tangent to the circle $x^2 + y^2 = 16$ for the points $x = 0$, $x = 2$, $x = -3$.

40. Find $D_s v$, where $v^2 = 2\,c\left(\frac{1}{s} - \frac{1}{a}\right)$.

41. $D_x y = \dfrac{x^2}{ax + b}$ and $u = \dfrac{a}{2}x^2 + bx$. Find $D_u y$.

42. Interpret geometrically the corollary of Art. 19 (f).

43. Find functions which give the following derivatives:

(a) $3\,x^2$; (b) $4\,x^3 - 1$; (c) $x^2 - x + 2$; (d) $-\dfrac{1}{x^2}$.

44. Deduce Theorem V, Art. 19, from Theorem IV by means of the substitution $\frac{u}{v} = w$, whence $D_x u = D_x (vw)$.

45. If the factor $(x - a)$ occurs n times in a function $f(x)$, show that it occurs $n - 1$ times in the derivative $D_x f(x)$, and from this principle deduce a method of finding whether an algebraic equation has multiple roots.

46. Show that a function of a function may be represented by a curve in space. [Echols' *Calculus*, Appendix, Note 2.]

CHAPTER IV

ELEMENTARY APPLICATIONS OF DERIVATIVES

24. Slope of a curve. The equations of the curves thus far discussed have all been given in the form

$$y = f(x), \quad \text{or } F(x, y) = 0. \tag{1}$$

In some cases it is more convenient to express both x and y in terms of a third variable; thus

$$x = \phi(t), \quad y = \psi(t), \tag{2}$$

where $\phi(t)$, $\psi(t)$ are single-valued functions of the variable t. We call these equations the *parametric equations* of the curve. As examples of parametric equations of curves, we have for the ellipse the equations

$$y = b \sin \theta, \quad x = a \cos \theta,$$

and for the cycloid

$$y = a\ (1 - \cos \theta)$$
$$x = a\ (\theta - \sin \theta).$$

We may pass from the parametric form of expression to that given by equation (1) by eliminating the common variable.

In whichever form the equation of the curve is given, we have

$$D_x y = \tan \phi, \tag{3}$$

where ϕ denotes the acute angle between the tangent to the curve and the positive direction of the X-axis. To determine $\tan \phi$ from the parametric equations of the curve, we have by Art. 23

$$D_x y = \frac{D_t y}{D_t x} = \tan \phi. \tag{4}$$

Tan ϕ has been defined as the slope or gradient of the tangent to the curve. It may also be called the *slope of the curve,* for the direction of the tangent to the curve is also the direction of the curve at the point of tangency. *The slope of the curve, therefore, is given by the value of the derivative $D_x y$.*

The value of the derivative at any point gives us still other properties of the curve. Since the direction of the tangent is identical with the direction of the curve at the point of tangency, it follows that so long as ϕ is a positive angle the ordinates of the curve are increasing as the abscissa increases. When, however, ϕ is positive, $\tan \phi$, and hence $D_x y$, is positive. For similar reasons when ϕ is negative, the ordinates of the curve are decreasing as the abscissa increases. But, in this case, $\tan \phi$, and hence $D_x y$, is negative. If the value of $D_x y$ becomes zero for any value of x, then $\tan \phi$ is zero and ϕ must be either zero or π; that is, the tangent to the curve at this point is parallel to the axis of X. This occurs whenever the curve has a turning-point, that is, a point where the ordinates cease to increase and begin to decrease, or vice versa. If, for any value of x, $D_x y$ becomes infinite, the value of ϕ is then $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, that is, the tangent to the curve at that point is perpendicular to the axis of X. We may summarize these results as follows:

At any point of the curve $y = f(x)$, the ordinate increases or decreases with increasing x according as $D_x y$ is positive or negative. If $D_x y$ is zero for any value of x, then at that point the tangent to the curve is parallel to the axis of abscissas. If it becomes infinite for any value of x, the tangent to the curve is perpendicular to the axis of abscissas at that point.

Ex. 1. Investigate the curve $y = \frac{1}{12}x^3 - x$ by means of its derivative.

Differentiating, we get

$$D_x y = \frac{3x^2}{12} - 1 = \frac{x^2}{4} - 1 = \frac{(x+2)(x-2)}{4}.$$

For values of $x > 2$ or $x < -2$, $D_x y$ is positive, and consequently y increases with x. For values of x between 2 and -2, $D_x y$ is negative, and y therefore decreases as x increases. For $x = 2$ and for $x = -2$, $D_x y = 0$; hence at these points the tangent to the curve is parallel to the X-axis. The curve is shown in Fig. 16.

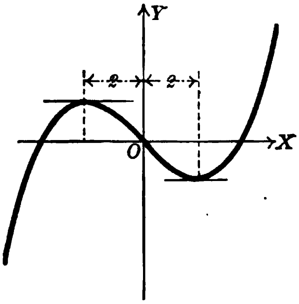

Fig. 16.

Ex. 2. The equation of van der Waals,

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT,$$

gives the relation between the pressure p, volume v, and absolute temperature T

of certain gases. By giving T various fixed values, we get a series of constant-temperature curves (isothermals) by means of which the behavior of the gas may be studied. Let us study the nature of these curves through the derivative.

To get the derivative D_vp, we write the equation in the form

$$p = \frac{RT}{v-b} - \frac{a}{v^2}. \tag{a}$$

Now, remembering that T is fixed and consequently RT is a constant, we have

$$D_vp = -\frac{RT}{(v-b)^2} + \frac{2a}{v^3}. \tag{b}$$

For convenience, we may write

$$D_vp = -\frac{1}{(v-b)^2}\left[RT - \frac{2a(v-b)^2}{v^3}\right]. \tag{c}$$

For carbon dioxide the following values are given for the constants: $R = 0.00369$, $a = 0.00874$, $b = 0.0023$. The values assigned to T may lie between 250 and 390. Since the factor $\frac{1}{(v-b)^2}$ is intrinsically positive, it follows that D_vp is negative or positive, according as RT is greater or less than $\frac{2a(v-b)^2}{v^3}$. This last term is small compared with RT for values of $v > 1$; but for small values of v, say for .005 to .01, the term becomes relatively large and may exceed RT if T is sufficiently small. As v is taken still smaller and finally approaches b, the fraction again becomes small. Hence for the smaller values of T, it follows that for values of v near b D_vp is negative. As v increases, a point is reached where

$$\frac{2a(v-b)^2}{v^3} = RT,$$

and here $D_vp = 0$. For larger values of v,

$$\frac{2a(v-b)^2}{v^3} > RT$$

and D_vp is positive.

However, as v increases still more, the fraction $\frac{2a(v-b)^2}{v^3}$, after reaching a maximum value, begins to decrease and again becomes equal to RT, whence D_vp is again zero. For all larger values of the derivative is negative.

The general form of the curve is shown at (a), Fig. 17. The curve (b) shows the form when T is taken so large that D_vp cannot take a positive value.

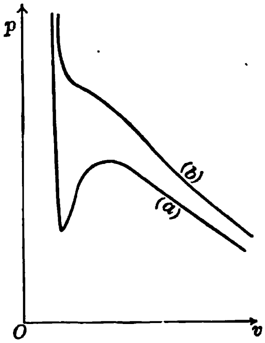

Fig. 17.

Branches of the curve may be found lying below the v-axis. These have, however, no physical interpretation.

EXERCISES.

1. Investigate the following curves by means of their derivatives.

(a) $y = x^3 - 3x^2 + 6x$. (b) $y^2 = 8x - 10$. (c) $y = \frac{2a}{x}$.

2. For what values of x does the function $x^2 + \frac{1}{x}$ increase with x? $> \sqrt[3]{}$ For what values does it decrease as x increases? $< \sqrt[3]{2}$

3. Investigate van der Waals' equation for the following values of T, using the constants previously given. (a) $T = 270$. (b) $T = 300$. It will be well to draw the curves carefully on cross-section paper.

4. The equation

$$\tau = 53.6\, p^{\frac{1}{5}} - 35.7$$

gives very closely the temperature τ of steam as a function of its pressure p. (Here τ is in degrees F., and p in pounds per square foot.) Show how the derivative $D_p\tau$ changes as the pressure increases, and sketch roughly the curve

$$\tau = f(p).$$

5. Discuss by means of the derivative $D_v H$ the equation

$$H = \frac{v}{825}\left(T - \frac{wv^2}{g}\right). \qquad \text{(See Art. 4, Ex. 6.)}$$

Find an expression for the speed beyond which the power decreases as the speed increases.

6. A cylindrical vessel with one end open is to hold 300 cu. in. The superficial area A of this vessel (cylindrical wall plus one base) is a function of the radius of the base. Deduce the equation and examine it by means of the derivative. Interpret the results.

7. Given the continuous curve $y = f(x)$. Show by means of the graph that for positive values of Δx, Δy is positive or negative according as the function is increasing or decreasing at a given point. By considering the limit $\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x}$ deduce the general law given on p. 59.

25. Derived curves. The derivative of a function is also, in general, a function of the independent variable, and may be represented by a graph in the same manner and under the same conditions as the original function. In what follows we shall restrict our discussion to those functions whose derivatives can be so represented, unless otherwise stated. This curve, whose equation is $y = f'(x)$, is known as the *derived curve*. The principles developed in the last article enable us to establish certain general properties of derived curves. Thus the graph of $y = f'(x)$ crosses the axis of x at those points for which the original curve has a turning-point. Moreover, we can tell whether the derived curve crosses the X-axis from above to below or vice versa; for we know that as x increases, $f'(x)$ is positive or negative according as $f(x)$ increases or decreases. Hence, if the values of $f(x)$ are increasing as the turning-point is approached, and decreasing after it is passed, as at point C, Fig. 18, the values of $f'(x)$ pass from positive to negative, and the derived curve passes from the upper to the lower side of the axis, as shown at point C'. At the turning-point A, on the other hand, $f(x)$ changes from a decreasing to an increasing function and $f'(x)$ passes from negative to positive values; that is, the derived curve passes from the lower to the upper side of the X-axis at the point A'.

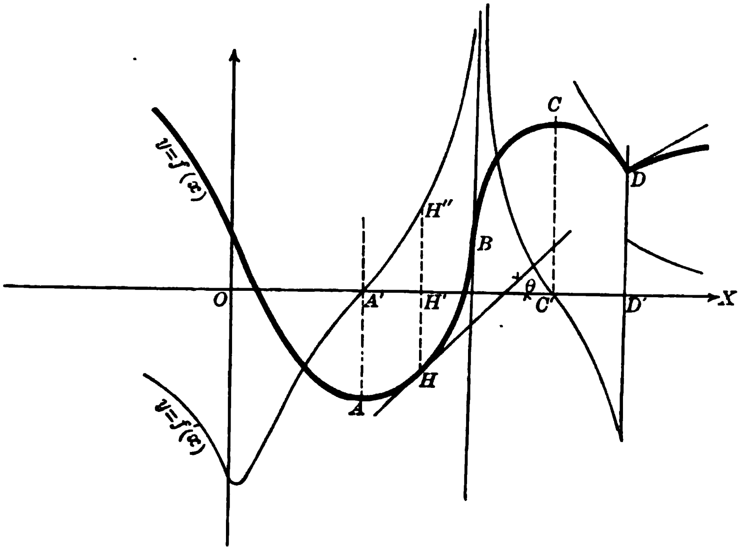

Fig. 18.

For those functions which are to be considered in the present volume the derived curve is, in general, a continuous curve. We shall meet, however, two exceptional cases in which it becomes discontinuous.

1. When the tangent to the curve $y = f(x)$ becomes perpendicular to the X-axis, as at B, Fig. 18. Here the value of $f'(x)$ becomes infinite and the derived curve has infinite branches.

2. When the curve $y = f(x)$ has an angular point as at D, Fig. 18. At such a point, the limiting position of the tangent as the point of tangency approaches D from the left is different from the limiting position as the point of tangency approaches D from the right. Hence, the value of $f'(x)$ takes a sudden jump as shown at D', and the derived curve has a discontinuity.

The ordinates of the derived curve are the successive values of D_xy of the original curve. Since D_xy measures the slope of the tangent to the original curve, it follows that for any value of x the ordinate of the derived curve measures the slope of the original curve. Thus at the point H, Fig. 18, where $\theta = \frac{\pi}{4}$, $\tan \phi = 1$, and ordinate $H'H'' = 1$.

It is evident that the slope of the curve $y = f(x)$ is independent of the position but not of the direction of the X-axis. We can shift the curve in the direction of the Y-axis, Fig. 18, and the values of $f'(x)$ will remain unchanged. In other words, the curves $y = f(x)$ and $y = f(x) + c$, where c is any constant, have the same derived curve (see Art. 19 (d)).

Ex. Draw the curve $y = \frac{x^3}{12} - x$ and its derived curve.

Differentiating, we get

$$D_xy = \frac{x^2}{4} - 1.$$

For various values of x, the ordinates representing $f(x)$ and $f'(x)$ are as follows :

x	$\pm 5,$	$\pm 4,$	$\pm 3,$	$\pm 2,$	$\pm 1,$	0
$f(x)$	$\pm 5\frac{5}{12},$	$\pm 1\frac{1}{3},$	$\pm \frac{3}{4},$	$\pm 1\frac{1}{3},$	$\pm \frac{11}{12},$	0
$f'(x)$	$5\frac{1}{4},$	$3,$	$1\frac{1}{4},$	$0,$	$-\frac{3}{4},$	$-1.$

Using these values, we get the curves shown in Fig. 19.

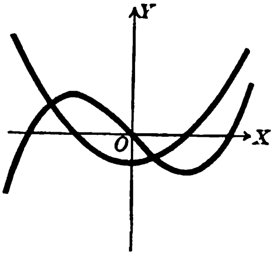

Fig. 19.

EXERCISES.

1. Draw the graph of each of the functions given in the first exercise of Art. 24, and also the derived curve of each.

2. Draw several curves, and by observing the variation in the slope and the turning-points sketch in approximately the derived curves.

3. Draw a curve at random, and by observing the variation of the ordinate draw roughly the curve of which the first curve is the derived curve.

26. Equation of the tangent and of the normal to a curve. Let (x_1, y_1) be any point on the curve whose equation is

$$y = f(x).$$

As we have already seen, the slope of the tangent to the curve at the point (x_1, y_1) is given by substituting the value x_1 for x in the derived function $f'(x)$, that is, by $f'(x_1)$. From analytic geometry, we know that the equation of the tangent at any point (x_1, y_1) on the curve is

$$y - y_1 = m(x - x_1),$$

where m denotes the slope of the tangent at the point in question. Replacing m by $f'(x_1)$, we have for the *equation of the tangent* to the curve in terms of the derivative

$$y - y_1 = f'(x_1)(x - x_1). \tag{1}$$

The normal to a curve is perpendicular to the tangent to the curve at the point of tangency. The condition that one straight line shall be perpendicular to another is that the slope of the one shall be the negative reciprocal of that of the other. It follows that the *equation of the normal* to the given curve at the point (x_1, y_1) is

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1). \tag{2}$$

Because of the relation

$$D_y x = \frac{1}{D_x y} = \frac{1}{f'(x)}, \qquad \text{(Art. 23.)}$$

the student will observe that frequently the slope of the normal can be most conveniently obtained by substituting $y = y_1$ in $D_y x$.

Ex. 1. Find the equations of the tangent and normal to the curve $y = x^{\frac{3}{2}}$ at the point (4, 8).

$$f'(x) = \tfrac{3}{2} x^{\frac{1}{2}},$$

and for $x = 4$, $\qquad f'(4) = \tfrac{3}{2} \times 2 = 3.$

Substituting this value of $f'(x)$ in (1), we have for the equation of the tangent

$$y - 8 = 3(x - 4),$$

or $\qquad 3x - y = 4.$

At the given point, $\dfrac{1}{f'(x_1)} = \tfrac{1}{3}$. Hence the equation of the normal is

$$y - 8 = -\tfrac{1}{3}(x - 4),$$

or $\qquad x + 3y = 28.$

Ex. 2. Find the equations of the tangent and normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

at the point (x_1, y_1).

We have $$y = b\sqrt{1 - \frac{x^2}{a^2}},$$

and $$f'(x) = -\frac{b}{a^2} \cdot \frac{x}{\sqrt{1 - \frac{x^2}{a^2}}} = -\frac{b^2 x}{a^2 y}.$$

For the point (x_1, y_1) it follows that the equation of the tangent is

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1),$$

which by means of the relation

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

reduces readily to

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1.$$

For the equation of the normal, we have

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1),$$

or $$\frac{y - y_1}{y_1} = \frac{a^2}{b^2}\,\frac{x - x_1}{x_1}.$$

$y - 4 = (x - 2)$

$8y - 32$

$8y - 4$

EXERCISES.

Find the equations of the tangent and normal to the following curves at the points indicated.

1. The parabola $y^2 = 8x$, at (2, 4).
2. The hyperbola $xy = a^2$, at (x_1, y_1).
3. The cissoid $y^2 = \dfrac{x^3}{2a - x}$, at (a, a).
4. The circle $(x - 4)^2 + (y + 3)^2 = 25$, at (7, 1).

$y - 4$ $x - 4$

Find the equations of the tangent and normal to the following curves at the point (x_1, y_1).

5. $y^3 = ax^2$.
6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
7. $x^n + y^n = a^n$.
8. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.
9. $y = x^3 - \dfrac{1}{x}$.
10. $y = x^3 - 3x + 10$.

27. Lengths of tangent, normal; subtangent, subnormal. The length of that part of the tangent to a curve which lies between the point of tangency and the axis of abscissas is called the *length of the* *tangent.* The length of that part of the normal intercepted between the same point on the curve and the axis of abscissas is called the *length of the normal.* The projections of these lengths upon the axis of abscissas are called respectively the *subtangent* and the *subnormal.*

Thus in Fig. 20, let the tangent to the curve at the point P, having the coördinates x_1, y_1, meet the X-axis in the point A, and let the normal at the point P meet it at C. Let BP be a perpendicular let fall from P upon the X-axis. Then for the point P, PA is the length of the tangent, PC is the length of the normal, and AB and BC are the subtangent and subnormal respectively. Denote by ϕ the angle PAC between the tangent to the curve and the X-axis. The angle BPC is then equal to ϕ. Assuming the equation of the curve to be $y = f(x)$, we may obtain directly from the figure the following values for the lengths of tangent and normal, subtangent and subnormal. The details of the proof are left as an exercise for the student.

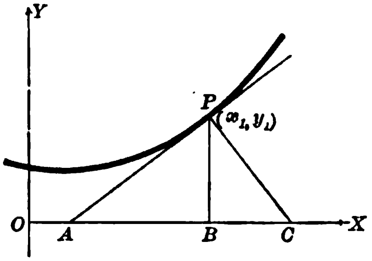

Fig. 20.

$$\textit{Length of tangent} = |y_1 \csc \phi| = \left|y_1\sqrt{1 + \left(\frac{1}{f'(x_1)}\right)^2}\right|. \tag{1}$$

$$\textit{Length of normal} = |y_1 \sec \phi| = |y_1\sqrt{1 + (f'(x_1))^2}|. \tag{2}$$

$$\textit{Subtangent} = |y_1 \cot \phi| = \left|\frac{y_1}{f'(x_1)}\right|. \tag{3}$$

$$\textit{Subnormal} = |y_1 \tan \phi| = |y_1 f'(x_1)|. \tag{4}$$

Instead of memorizing these formulas, the student will do well to so familiarize himself with the steps in their development that he can apply the fundamental principles involved to any special case.

Ex. Find the subtangent, subnormal, and lengths of tangent and normal for the curve $y^2 = \frac{x^3}{4}$ at the point (4, 4).

We have $$y = \tfrac{1}{2} x^{\frac{3}{2}},$$

whence $$D_x y = f'(x) = \tfrac{3}{4} x^{\frac{1}{2}},$$

and $$f'(4) = \tfrac{3}{4}\sqrt{4} = \tfrac{3}{2}.$$

Hence we have

$$\text{Subtangent} = \left| \frac{y_1}{f'(x_1)} \right| = \frac{4}{\frac{3}{2}} = 2\tfrac{2}{3}.$$

$$\text{Subnormal} = | y_1 f'(x_1) | = 4 \times \tfrac{3}{2} = 6.$$

$$\text{Length of tangent} = \left| y_1 \sqrt{1 + \left(\frac{1}{f'(x_1)}\right)^2} \right| = 4\sqrt{1 + (\tfrac{2}{3})^2} = \tfrac{4}{3}\sqrt{13}.$$

$$\text{Length of normal} = | y_1 \sqrt{1 + (f'(x_1))^2} | = 4\sqrt{1 + (\tfrac{3}{2})^2} = 2\sqrt{13}.$$

EXERCISES.

Find the subtangent, subnormal, length of tangent, and length of normal for each of the following curves at the points indicated.

1. $y^2 = 8x$, at $(2, 4)$.
2. $xy = a^2$, at (x_1, y_1).
3. $(x - 4)^2 + (y + 3)^2 = 25$, at $(7, 1)$.
4. $y = x^3 - 3x + 10$, at (x_1, y_1).
5. $y^3 = ax^2$, at (a, a).
6. $y^2 = \dfrac{x^3}{2a - x}$, at (a, a).
7. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, at (x_1, y_1).
8. Show that the subnormal to the parabola $y^2 = 2mx$ is constant.
9. Using the equations of the tangent and normal in Art. 26, derive formulas (1), (2), (3), and (4) by finding the intercepts of these lines on the X-axis.
10. Show that the length of the normal to the curve $x^2 + y^2 = a^2$ is constant.

28. Tan ψ, cot ψ. Let ρ, θ be the polar coördinates of any point on a curve. It is customary to assume θ as the independent variable. In order to determine the direction of the curve at any point, it is convenient to express the tangent and cotangent of the angle between the radius vector and the tangent to the curve in terms of ρ, θ, and their derivatives. Let us denote this angle by ψ.

In Fig. 21, let P be any point on the given curve, and let its polar coördinates be ρ, θ. Let θ take an increment $\Delta\theta$, then ρ will have a corresponding

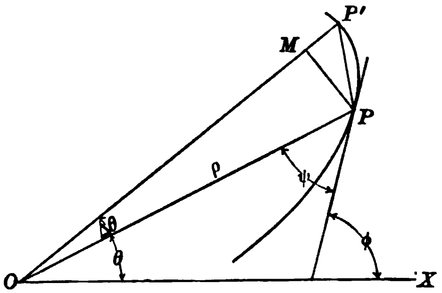

Fig. 21.

increment $\Delta\rho$. Corresponding to the coördinates $\theta + \Delta\theta$, $\rho + \Delta\rho$, we have a second point P'. Let MP be drawn perpendicular to OP'.

From the figure, we have

$$\begin{aligned} MP &= \rho \sin \Delta\theta, \\ OM &= \rho \cos \Delta\theta, \\ MP' &= OP' - \rho \cos \Delta\theta \\ &= \rho + \Delta\rho - \rho \cos \Delta\theta. \end{aligned}$$

Hence $$\tan MP'P = \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho \cos \Delta\theta}.$$

As $\Delta\theta \doteq 0$, the angle $MP'P$ approaches the angle ψ. Hence we have

$$\underset{\Delta\theta \doteq 0}{L} \tan MP'P = \underset{\Delta\theta \doteq 0}{L} \frac{\rho \sin \Delta\theta}{\Delta\rho + \rho\,(1 - \cos \Delta\theta)},$$

or $$\tan \psi = \underset{\Delta\theta \doteq 0}{L} \frac{\rho \dfrac{\sin \Delta\theta}{\Delta\theta}}{\dfrac{\Delta\rho}{\Delta\theta} + \rho \left(\dfrac{1 - \cos \Delta\theta}{\Delta\theta}\right)}.$$

But we have seen that

$$\underset{\Delta\theta \doteq 0}{L} \frac{\sin \Delta\theta}{\Delta\theta} = 1, \quad \underset{\Delta\theta \doteq 0}{L} \frac{1 - \cos \Delta\theta}{\Delta\theta} = 0; \qquad \text{(Arts. 11, 13.)}$$

hence, we have

$$\tan \psi = \frac{\rho}{D_\theta \rho} = \rho D_\rho \theta. \tag{1}$$

From (1) it follows at once that

$$\cot \psi = \frac{1}{\tan \psi} = \frac{1}{\rho} D_\theta \rho. \tag{2}$$

Ex. Let the equation of a curve be $\rho^2 = \dfrac{a}{\theta}$. Find the angle ψ for $\rho = 1$.

Writing the given equation in the form

$$\theta = \frac{a}{\rho^2} = a\rho^{-2},$$

we find $\quad D_\rho \theta = -2a\rho^{-3}$.

Hence $\quad \tan \psi = \rho D_\rho \theta = -2a\rho^{-2} = -2\dfrac{a}{\rho^2}$,

and $\quad \cot \psi = -\dfrac{\rho^2}{2a}$.

For $\quad \rho = 1$, $\tan \psi = -2a$,

whence $\quad \psi = \text{arc tan}\,(-2a)$.

29. Length of polar tangent and polar normal; polar subtangent, polar subnormal. Given a curve and the polar coördinates (ρ_1, θ_1) of any point P upon it. At the point P, Fig. 22, draw the tangent and the normal to the curve and through the pole O draw a perpendicular to the radius vector OP. Let this perpendicular cut the tangent and the normal in the points T and N respectively. PN is called the *length of the polar normal*, and PT the *length of the polar tangent*. The projections of these lines upon the perpendicular, namely, NO and OT, are called the *polar subnormal* and *polar subtangent*, respectively.

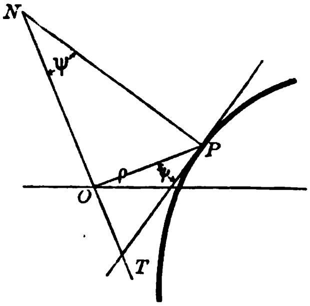

Fig. 22.

If the equation of the given curve is $\rho = F(\theta)$, we have from the figure

$$\text{Angle } PNO = \text{angle } OPT = \psi.$$

$$\textbf{\textit{Length of polar tangent}} = |\rho_1 \sec\psi| = \left|\rho_1\sqrt{1+\rho_1^2\left(\frac{1}{F'(\theta_1)}\right)^2}\right|. \quad (1)$$

$$\textbf{\textit{Length of polar normal}} = |\rho_1 \csc\psi| = \left|\sqrt{\rho_1^2+(F'(\theta_1))^2}\right|. \quad (2)$$

$$\textbf{\textit{Polar subtangent}} = |\rho_1 \tan\psi| = \left|\frac{\rho_1^2}{F'(\theta_1)}\right|. \quad (3)$$

$$\textbf{\textit{Polar subnormal}} = |\rho_1 \cot\psi| = |F'(\theta_1)|. \quad (4)$$

It will be observed that in the case of rectangular coördinates, the line upon which the tangent and normal were projected to determine the subtangent and subnormal was the X-axis, while in polar coördinates it is not the initial line, but a line through the pole perpendicular to the radius vector.

Ex. Find the lengths of the polar tangent and polar normal, also the subtangent and subnormal of the curve $\rho^2 = \frac{a^2}{\theta}$.

Putting the equation in the form $\rho = a\theta^{-\frac{1}{2}}$,

we get $$D_\theta\rho = -\tfrac{1}{2}a\theta^{-\frac{3}{2}}; \text{ hence, } F'(\theta_1) = -\frac{a}{2\theta_1^{\frac{3}{2}}}.$$

Substituting this value in the proper formulas, we have

$$\text{Length of polar tangent} = \frac{a}{\sqrt{\theta_1}}\sqrt{1 + 4\,\theta_1^2}.$$

$$\text{Length of polar normal} = \frac{a}{\sqrt{\theta_1}}\sqrt{1 + \frac{1}{4\,\theta_1^2}}\cdot$$

$$\text{Polar subtangent} = 2\,a\sqrt{\theta_1}.$$

$$\text{Polar subnormal} = \frac{a}{2\,\theta_1^{\frac{3}{2}}}\cdot$$

EXERCISES.

1. Find general expressions for the length of tangent, subtangent, length of normal and subnormal of the spiral $\rho = a\theta$. Show also that $\tan\psi = \theta$.

2. Find the same for the curves

$$\text{(a) } \rho = a\theta^2. \quad \text{(b) } \rho = 1 + \frac{1}{\theta}\cdot \quad \text{(c) } \rho^2 = a\theta + b\theta^2.$$

3. Find $\tan\psi$ for the curves of Ex. 2.

4. Show that the polar subtangent to the curve $\rho\theta = a$ is of constant length. Trace the curve.

30.. Multiple roots. Given an integral rational function, that is, a function of the form

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n\,. \tag{1}$$

From algebra, we know that if any number a is a root of $f(x) = 0$, $(x - a)$ is a factor of $f(x)$. Let this root occur k times in $f(x) = 0$. We may then write

$$f(x) = (x - a)^k \cdot \phi(x), \tag{2}$$

where $\phi(x)$ does not contain the factor $(x - a)$. Differentiating, we have

$$f'(x) = k\,(x - a)^{k-1} \cdot \phi(x) + (x - a)^k \cdot \phi'(x). \tag{3}$$

From this we see that if the factor $(x - a)$ occurs k times in a function, it occurs $(k - 1)$ times in the derived function. Consequently, any multiple root of $f(x) = 0$ is also a root of $f'(x) = 0$ of dimension one less. All of the multiple roots of $f(x) = 0$ are determined by the common factors of $f(x)$ and $f'(x)$. Conversely, any factor $(x - a)$ which is common to $f(x)$ and $f'(x)$ must of necessity appear to degree one higher in the function than in its derivative, and hence only multiple roots are to be found by equating to zero those common factors. We have therefore the following statement:

To find the multiple roots of $f(x) = 0$, equate to zero the greatest common divisor of $f(x)$ and $f'(x)$. Each root of the equation thus obtained is a multiple root of order one higher of $f(x) = 0$.

Ex. 1. Examine the equation $3x^4 - 25x^3 + 63x^2 - 27x - 54 = 0$ for multiple roots.

$$f(x) = 3x^4 - 25x^3 + 63x^2 - 27x + 54,$$

whence

$$f'(x) = 12x^3 - 75x^2 + 126x - 27.$$

The greatest common factor of $f(x)$ and $f'(x)$ is $(x - 3)^2$. Hence 3 is a double root of $f'(x) = 0$ and a triple root of $f(x) = 0$. The roots of $f(x) = 0$ are found to be $3, 3, 3, -\frac{2}{3}$.

Ex. 2. Find the relation that must exist between q and r that the equation $x^3 - qx + r = 0$ may have a double root.

We have

$$f(x) = x^3 - qx + r,$$
$$f'(x) = 3x^2 - q.$$

If α is a double root of $f(x) = 0$, it is also a root of $f'(x) = 0$; hence, we must have simultaneously

$$\alpha^3 - q\alpha + r = 0,$$
$$3\alpha^2 - q = 0.$$

Eliminating α, we get $4q^3 = 27r^2$ as the required relation.

EXERCISES.

1. Find the multiple roots of the following equations:

(a) $x^3 - x^2 - 8x + 12 = 0.$
(b) $2x^5 - 7x^4 + 8x^3 - 2x^2 - 2x + 1 = 0.$
(c) $x^4 - 6x^3 + 9x^2 + 4x - 12 = 0.$

2. Find the condition that the equation $x^3 + px^2 + r = 0$ shall have a double root.

31. Rolle's theorem. The following proposition known as Rolle's theorem will be of frequent use.

*If $f(x)$ and $f'(x)$ are single-valued and continuous for all values of x from $x = a$, to $x = b$, and if $f(a) = f(b) = 0$, then $f'(x)$ vanishes for at least one value of x between a and b.**

Either $f(x)$ has a constant value zero for all values of x between a and b, or it varies with x. In the first case $f'(x)$ is zero for all values of x. In the second case, since

$$f(a) = f(b) = 0,$$

---

* Rolle's theorem is stated here in sufficiently general terms for our present purposes. The proof given holds for the theorem as stated. For a more general statement of the theorem and its proof, see Pierpont's *Theory of Functions*, Vol. I, p. 246.

$f(x)$ must at some point begin to increase and afterwards decrease, or vice versa. It must then have a turning-point for some particular value of x, say $x = x_1$, lying between a and b, since, by hypothesis, $f(x)$ is continuous.

Geometrically, Rolle's theorem means that if a continuous curve cuts the X-axis in two points $x = a$, $x = b$, and has a definite direction at every point in this interval (a, b), then at some intervening point, say $x = x_1$, the tangent to the curve is parallel to the X-axis. (Fig. 23.)

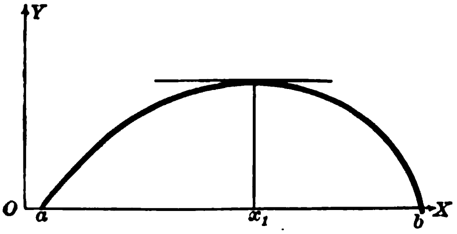

Fig. 23.

It is at once evident that instead of the condition $f(a) = f(b) = 0$, we might have $f(a)$ and $f(b)$ equal to any constant so long as they are equal to each other. The argument remains precisely the same in the two cases.

EXERCISES.

1. By Rolle's theorem show that at least one real root of the equation $f'(x) = 0$ lies between any two real roots of the equation $f(x) = 0$.

2. From Ex. 1 show that if two roots of $f(x) = 0$ are equal, one root of $f'(x) = 0$ coincides with them. Give a geometric illustration of this statement.

3. From Ex. 1, k real roots of $f(x) = 0$ have at least $k - 1$ real roots of $f'(x) = 0$ lying between them. Making use of this fact, deduce the method of Art. 30 for finding multiple roots. Illustrate by a figure.

32. Law of the mean. By the use of Rolle's theorem, we may now deduce one of the most fundamental theorems of the differential calculus, known as the *law of the mean*, or the theorem of mean value. This theorem may be stated as follows:

Let $f(x)$ and $f'(x)$ be single-valued and continuous functions of x in the interval $a \leqq x \leqq b$. Then there exists at least one value x_0 of x for which

$$\frac{f(b) - f(a)}{b - a} = f'(x_0), \quad a < x_0 < b.$$

Without loss of generality, we may assume $f(a) < f(b)$. Geometrically $\frac{f(b) - f(a)}{b - a}$ gives the slope of the line AB, Fig. 24. Let $y = f(x)$ be represented by the curve $APDB$, passing through the points A and B; $f'(x_0)$ is then the slope of the tangent to this curve at the point D. The theorem asserts geometrically that there exists at least one point D of the curve at which the tangent is parallel to the secant line AB.

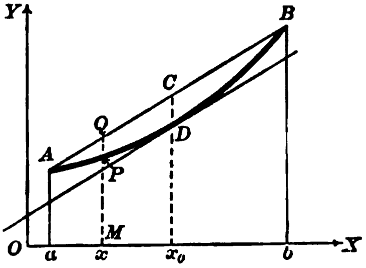

Fig. 24.

The theorem may be established analytically as follows: The equation of the line AB is

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a). \tag{1}$$

The distance of any point P on the curve from the line AB, when measured along the ordinate through P, is given by

$$\begin{aligned} \psi(x) &= PQ = MQ - MP \\ &= \frac{f(b) - f(a)}{b - a}(x - a) + f(a) - f(x). \end{aligned} \tag{2}$$

If the tangent to the curve $y = f(x)$ at some point, is to be parallel to AB, then $\psi(x)$ for the corresponding value of x, must have a turning-point. Moreover, $\psi(x)$ is a function which satisfies all of the requirements of Rolle's theorem. At some point, say $x = x_0$, of the interval (a, b), we have therefore

$$\psi'(x_0) = \frac{f(b) - f(a)}{b - a} - f'(x_0) = 0, \tag{3}$$

whence

$$\frac{f(b) - f(a)}{b - a} = f'(x_0), \tag{4}$$

as the theorem requires.

The theorem may be stated in another form which is sometimes convenient. The fact that x_0 lies between a and b can be expressed by the relation

$$x_0 = a + \theta(b - a),$$

where θ is some number between 0 and 1. We may also put

$$b - a = h.$$

Equation (4) then takes the form

$$\frac{f(a+h) - f(a)}{h} = f'(a + \theta h). \tag{5}$$

The fraction $\frac{f(b) - f(a)}{b - a}$ evidently measures the average rate of increase of the function in the interval $b - a$. Thus, suppose s_1 and s_2 denote distances traversed by a moving point in the times t_1 and t_2 respectively; the fraction $\frac{s_2 - s_1}{t_2 - t_1}$ gives the average speed of the point for the time interval $t_2 - t_1$, and the law of the mean asserts that at some instant within this interval the actual instantaneous speed of the point is equal to the average or mean speed for the whole interval.

Ex. Show geometrically that Rolle's theorem is a special case of the law of the mean.

33. Approximate value of small errors. When a physical measurement is used as a basis of computation, an error in the measurement gives rise to an error in the calculated result. The magnitude of the error in the result may be found approximately by the application of the law of the mean as follows:

Let x denote the measured quantity and Δx the error in the measurement. Then if $y = f(x)$, the corresponding error in the function is

$$\Delta y = f(x + \Delta x) - f(x). \tag{1}$$

From the law of the mean, (1) may be written

$$\Delta y = f'(x + \theta \Delta x) \cdot \Delta x, \tag{2}$$

where θ is a number between 0 and 1. Provided $f'(x)$ is continuous, we have approximately therefore

$$\Delta y = f'(x) \cdot \Delta x. \tag{3}$$

Frequently the ratio $\frac{\Delta y}{y}$, the *relative error*, is desired. This ratio is given approximately by the equation

$$\frac{\Delta y}{y} = \frac{f'(x)}{f(x)} \Delta x \tag{4}$$

Ex. The volume V of a sphere is to be determined from a measurement of its diameter d. Determine the relative error in the calculated volume due to an error in the measurement.

Since $$V = \tfrac{1}{6}\pi d^3,$$

we have $$D_d V = f'(d) = \tfrac{1}{2}\pi d^2,$$

whence approximately $$\Delta V = \tfrac{1}{2}\pi d^2 \Delta d,$$

and $$\frac{\Delta V}{V} = \frac{\frac{1}{2}\pi d^2}{\frac{1}{6}\pi d^3}\Delta d = 3\frac{\Delta d}{d}.$$

Hence, the relative error in the volume is three times that of the measurement.

EXERCISES.

1. The side of a square is 8 inches with a possible error of 0.01 inch. Find the possible error in the area of the square.

2. Find the relative error in the area of a circle due to an error Δr in the measurement of its radius r.

3. The heat developed in a circuit by an electric current is given by the formula

$$Q = ki^2Rt,$$

in which k is a constant, i is the strength of the current (in amperes), R the resistance of the circuit, and t the time.

(a) Show that an error in the measurement of i makes double the error in Q that the same relative error in R or t would make.

(b) Find the relative error in Q due to an error of 0.1 per cent in i.

4. Apply the principles of this section to three physical laws in which results are calculated from observed data.

34. Speed and acceleration. Let a point move in a given path, straight or curved, and let s denote the variable distance of the point measured along the path from some fixed origin O. Assuming continuous motion from O, the distance s evidently depends upon the time t; that is, $s = f(t)$. In a time interval Δt the point moves over an element of path of length Δs, and the ratio $\frac{\Delta s}{\Delta t}$ is defined as the *mean speed* of the point for this interval. The limiting value of this quotient as the time interval Δt approaches zero gives the *instantaneous speed* at the beginning of the interval. Denoting this by v, we have therefore

$$v = \underset{\Delta t \doteq 0}{L} \frac{\Delta s}{\Delta t} = D_t s; \tag{1}$$

that is, *the speed of a moving point is the time-derivative of the space traversed.*

The words " speed " and "velocity" are often used as synonyms. An important distinction is, however, frequently made. The term " speed " is used to indicate merely the rate of motion in the path irrespective of direction. Speed, therefore, has only magnitude. The term " velocity " carries with it the additional notion of direction, hence to specify a velocity both magnitude and direction are required.

The speed v is, in general, a function of the time t. If Δv denotes the change of speed between two points, then the quoti $\frac{\Delta v}{\Delta t}$ is defined as the *average tangential acceleration* * between th points; and the limit of this quotient as $\Delta t \doteq 0$ is defined as th *instantaneous tangential acceleration.* Denoting this by a, we have

$$a = \underset{\Delta t \doteq 0}{L} \frac{\Delta v}{\Delta t} = D_t v; \tag{2}$$

that is, *the tangential acceleration is the time-derivative of the speed.*

The ordinary unit of speed is the foot per second, that of acceleration the foot per second per second. These are abbreviated to ft./sec. and ft./sec.^{2} respectively.

Ex. 1. In a certain motion the space described is expressed as a function of the time by the following equation:

$$s = at^2 + bt + c. \tag{3}$$

For the speed, we have

$$v = D_t s = 2\,at + b, \tag{4}$$

and for the tangential acceleration, we obtain

$$a = D_t v = 2\,a. \tag{5}$$

Ex. 2. In the case of bodies falling in vacuo, $s = \frac{1}{2}gt^2$, where $g = 32.2$ nearly. Find the speed at the end of 5 seconds, also the acceleration.

$$v = D_t s = gt = 32.2\,t, \tag{6}$$

hence for $t = 5$ $\qquad v = 32.2 \times 5 = 161.$

$$a = D_t v = g = 32.2. \tag{7}$$

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* Tangential acceleration is the acceleration in the direction of the motion, that is, tangent to the path. In the case of rectilinear motion, this is the only acceleration; but in the case of a point moving in a curve, there is another acceleration perpendicular to the tangent.

35. Speed curves; acceleration curves. If we take t the abscissa and lay off the corresponding values of s determined by the relation $s = f(t)$ as ordinates, we shall get a curve, as m, Fig. 25. If now the successive values of the speed v determined from the relation

$$v = D_ts = f'(t) \tag{1}$$

be laid off as ordinates, a second curve m_1 is obtained. This is evidently the derived curve of m. In the same way the successive values of the tangential acceleration

$$a = D_tv \tag{2}$$

may be laid off, giving the curve m_2. It is clear that m_2 is the derived curve of m_1. From the properties of derived curves, Art. 25, the ordinate m_1 is equal to the slope of m, and the ordinate of m_2 is the slope of m_1. Whenever m_1 has a turning-point, as at A, m_2 crosses the axis of abscissas as at B, and the acceleration a changes from positive to negative, or *vice versa*.

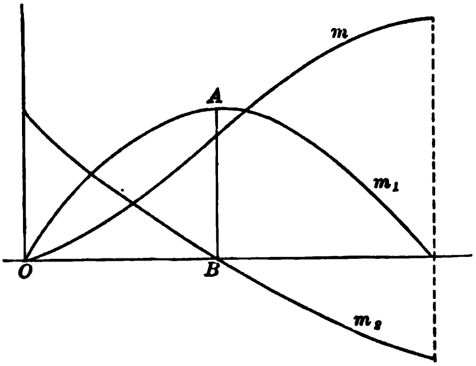

Fig. 25.

The curves m_1 and m_2 are called respectively *speed and acceleration curves* on a time base.

Frequently it is more convenient to use the distance s as the abscissa for the speed curve. We have in this case

$$v = \phi(s) \text{ and } s = f(t). \tag{3}$$

To get an expression for the tangential acceleration a, we use the formula for the derivative of a function of a function; thus,

$$a = D_t v = D_s v \cdot D_t s. \qquad \text{(Art. 21.)}$$

But $$D_t s = v;$$

hence $$a = v D_s v.$$

The curve in Fig. 26 is a speed curve on a distance base and its equation is $v = \phi(s)$. From Art. 27, the product $v D_s v$ gives the length of the subnormal, RS, for any point P. Hence, *if a speed curve is drawn on a distance base, the subnormal measures the acceleration; if it is drawn on a time base, the slope measures the acceleration.*

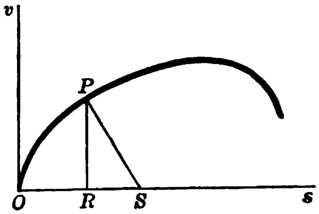

Fig. 26.

Ex. For falling bodies the speed is expressed as a function of the distance by means of the equation $v^2 = 2gs$. Hence the speed curve on a distance base is a parabola. The acceleration is

$$v D_s v = v \cdot D_s(2gs)^{\frac{1}{2}} = g. \qquad (4)$$

36. Angular speed and acceleration. If a body rotates about a fixed axis, any given point of it, not on the axis, moves in a circle whose center lies on this axis, and whose plane is perpendicular to the axis. Let O be the center and PAB the circular path of the point, Fig. 27. During the motion of the point from P to A, the radius OA sweeps over an angle θ, and in the additional interval of time Δt required for the motion from A to B it sweeps over the angle $AOB = \Delta\theta$. The ratio $\frac{\Delta\theta}{\Delta t}$ is defined as the *mean angular speed* of the body between the positions OA and OB, and the limit of this ratio as B is made to approach A is defined as the *instantaneous angular speed* for the position OA of the radius. Denoting this angular speed by ω, we have

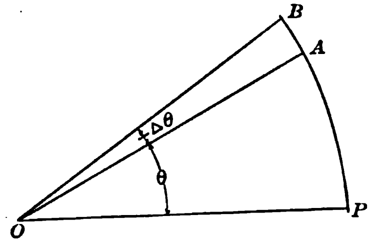

Fig. 27.

$$\omega = \underset{\Delta t \to 0}{L} \frac{\Delta\theta}{\Delta t} = D_t\theta; \qquad (1)$$

that is, *the angular speed is the time-derivative of the angle swept over.*

The angular speed may be constant or variable. If variable, the increment between two positions, say A and B, may be denoted by $\Delta\omega$, and the ratio $\frac{\Delta\omega}{\Delta t}$ is called the *mean angular acceleration* between the positions in question. The limit of this mean value as the two chosen positions are made to approach each other is the *instantaneous angular acceleration*, which is denoted by α. We have then

$$\alpha = \underset{\Delta t \doteq 0}{L} \frac{\Delta\omega}{\Delta t} = D_t\omega; \qquad (2)$$

that is, *the angular acceleration is the time-derivative of the angular speed.*

With angles measured in radians, the unit of angular speed is the radian per second (rad./sec.), and that of angular acceleration is the radian per second per second (rad./sec.^{2}).

The relation between the angular speed of a rotating body and the linear speed of any point of the body in its circular path is readily derived. Referring to Fig. 27, AB is an element Δs of the circular path of P. Denoting the radius OA by r, we have therefore

$$\Delta s = r\Delta\theta,$$

whence

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t},$$

and finally

$$D_t s = rD_t\theta; \qquad (3)$$

that is, *the speed of any point of a body rotating about a fixed axis is the product of the distance of the point from the axis and the angular speed of the body.*

A similar law holds for tangential acceleration. Thus from (3)

$$v = r\omega,$$

whence

$$D_t v = rD_t\omega,$$

or

$$a = r\alpha; \qquad (4)$$

that is, *the tangential acceleration of the point is the angular acceleration about the fixed axis multiplied by the distance from the point to the axis.*

EXERCISES.

1. If a body is projected vertically upward with a speed of v_0 feet per second, the space traversed in t seconds from the instant of projection is given by the equation

$$s = v_0 t - \tfrac{1}{2} g t^2.$$

(a) Find expressions for the speed and acceleration at the time t. (b) Take $v_0 = 220$ and $g = 32.2$, and find the speed and acceleration at the end of 3 seconds; (c) at the end of 10 seconds.

2. In Ex. 1, find the whole time occupied by the body rising and falling also the height to which it rises.

Suggestion: Make $s = 0$ and solve for t.

3. Draw speed and acceleration curves for the motion described in Ex. 1: (a) on a time base; (b) on a distance base.

4. The following values of distances and times were taken from the speed chart of a railway test car. Take the times as abscissas, the distances as ordinates, and plot the distance-time curve. From this draw roughly the speed curve, and from the speed curve draw approximately the acceleration curve.

Sec.	Feet.	Sec.	Feet.
5	30	35	585
10	88	40	715
15	160	45	840
20	250	50	970
25	350	55	1130
30	470		

5. When a body moves in a straight line under the influence of an attractive force which varies as the inverse square of the distance, the motion is given by the equation

$$v^2 = \frac{k}{s},$$

in which k is a constant. Show that the acceleration is $-\dfrac{k}{2s^2}$.

6. Take a line whose length is the initial distance s_0 and on this as a distance base draw the speed and acceleration curves for Ex. 5.

7. In the test of an electric locomotive the following speeds were observed:

Time from start, seconds	10	20	30	40	50	60	70	80	90	100	110	120	130	140	145
Speed, miles per hour	10.4	19.2	24.5	28.0	32.0	34.2	36.8	38.0	39.5	41.0	40.2	37.0	29.0	16.0	0

Using the times as abscissas, draw the speed curve and sketch roughly the acceleration curve.

8. The angle (in radians) through which a given rotating body turns, starting from rest, is given by the equation

$$\theta = 5t^2.$$

Find the angular speed and angular acceleration at the end of 4 seconds.

9. If the angle is given by the equation

$$\theta = 112\,t - 16\,t^2,$$

find (a) the speed and acceleration at the end of 2.5 seconds; (b) the time that elapses before the body comes to rest.

10. Derive expressions for ω and α from each of the following relations between θ and t:

(a) $\theta = at - bt^3$. (b) $\theta = at^{\frac{1}{2}}$. (c) $\theta = a + bt + ct^2$.

37. Miscellaneous applications. In all branches of physics and chemistry the notion of the derivative is repeatedly encountered. In the case of a moving point, the term "speed" is used for the time-derivative of the distance traversed. In chemistry, likewise, we have the same term used for other time-derivatives; thus the speed of reaction and the speed of solution are such derivatives. In physics we meet with a great number of derivatives expressing the rates of change of various physical magnitudes. The heating of substances, variations of pressure, density, and temperature, the variations in velocity and in energy, etc., are such changes. Some of these derivatives are discussed in the following paragraphs.

$D_T x$

(a) *Coefficients of expansion.* Let a rod or wire have unit length (1 foot or 1 yard) at some standard temperature, say 32° F. or 0° C. When heated the rod expands by an amount x depending upon the temperature. Thus, denoting the temperature by τ, the expansion is $x = f(\tau)$ and the new length of the rod is

$x = f(T)$

$$1 + x = 1 + f(\tau). \tag{1}$$

If now the temperature rises by an amount $\Delta\tau$, the length of the rod will increase by a corresponding amount Δx. The quotient $\frac{\Delta x}{\Delta\tau}$ is called the *average coefficient of linear expansion* for the interval $\Delta\tau$, and its limit as $\Delta\tau$ is made to approach zero is the *coefficient of linear expansion for the temperature* τ. Denoting this by C_1, we have

$$C_1 = \underset{\Delta\tau \doteq 0}{L} \frac{\Delta x}{\Delta\tau} = D_\tau x. \tag{2}$$

In most cases we may assume with sufficient approximation

$$x = f(\tau) = a\tau + b\tau^2,$$

whence

$$C_1 = a + 2\,b\tau; \tag{3}$$

that is, C_1 is itself a function of the temperature.

In the same way we may arrive at the coefficients of superficial and cubical expansion. Consider a cube with its edge having unit length at 0° C.

Assuming equal expansion in all directions, each edge at the temperature τ will have a length $1 + f(\tau)$. Hence the area of a face will be

$$A = [1 + f(\tau)]^2, \tag{4}$$

and the volume of the cube will be

$$V = [1 + f(\tau)]^3. \tag{5}$$

The derivatives $D_\tau A$ and $D_\tau V$ are the *coefficients of superficial and cubical expansion* respectively, and may be denoted by C_2 and C_3.

(b) *Specific heat.* A body, originally at some definite temperature τ_0, is heated and its temperature rises The heat Q absorbed by the body is in general a function of the final temperature τ; that is,

$$Q = f(\tau). \tag{6}$$

An increment of heat ΔQ causes a corresponding rise in temperature $\Delta\tau$, and the quotient $\frac{\Delta Q}{\Delta\tau}$ is called the *average specific heat* of the body for the interval $\Delta\tau$. The limit of this quotient as $\Delta\tau$ approaches zero is defined as the *specific heat at the temperture* τ. Denoting this by c, we have

$$c = \underset{\Delta\tau \doteq 0}{L} \frac{\Delta Q}{\Delta\tau} = D_\tau Q. \tag{7}$$

(c) *Compressibility and elasticity.* The volume v of a mass of fluid changes by the amount Δv due to a change of pressure Δp. The change of volume per unit volume is the quotient $\frac{\Delta v}{v}$. The limit of the ratio of this quotient to the change of pressure Δp, that is,

$$\underset{\Delta p \doteq 0}{L} \left(- \frac{\Delta v}{v \Delta p} \right) = - \frac{1}{v} D_p v \tag{8}$$

is defined as the *compressibility* of the fluid. The negative sign is given because change of volume Δv is a diminution.

The reciprocal of the compressibility is called the *volume elasticity* of the fluid. Denoting this latter by e, we have

$$e = - v \frac{1}{D_p v} = - v D_v p. \tag{9}$$

Ex. A gas expands according to the law $pv^k = C$. To find the volume elasticity, we have

$$p = Cv^{-k},$$

whence $$D_v p = - kCv^{-k-1},$$

and $$e = - v D_v p = kCv^{-k} = kp. \tag{10}$$

EXERCISES.

1. Regnault's experiments on the heating of various substances are represented by the following equations:

Ether,	$Q = 0.5290\,\tau + 0.000296\,\tau^2$.
Chloroform,	$Q = 0.2324\,\tau + 0.00005\,\tau^2$.
Bisulphide of carbon,	$Q = 0.2352\,\tau + 0.000082\,\tau^2$.

In each case Q denotes the heat required to raise the temperature of the substance from 0° C. to τ° C. For each substance find the specific heat at 20° C.

2. It is found by experiment that the volume of water which at 4° C. has unit volume is given by the equation

$$V = 1 + a(\tau - 4)^2,$$

where τ denotes the temperature of the water and $a = 0.00000838$. Find the coefficient of cubical expansion when $\tau = 0°$; also for $\tau = 20°$.

3. A gas expands at constant temperature (isothermally), following the law $pv = C$. Show that the volume elasticity is equal to the pressure p.

4. The electrical resistance R of a wire varies with the temperature τ of the wire, the relation being expressed by $R = f(\tau)$. (a) What is expressed by the derivative $D_\tau R$? (b) Find the expression for this derivative from Callendar's formula

$$R = R_0(1 + \alpha\tau + \beta\tau^2),$$

in which R_0, α, and β are constants.

5. (a) What derivative expresses the rate of the rise of temperature of a gas with respect to the pressure? (b) with respect to the volume? (c) What derivative gives the rate of change of the energy U of the gas with respect to the temperature τ?

6. The pressure of the atmosphere decreases as the distance from the earth's surface increases, and the rate of change of pressure with the height is proportional to the pressure. State this law in the form of an equation.

MISCELLANEOUS EXERCISES.

1. The equation of a curve is $4\,a^2y = \frac{1}{3}\,x^3 - 2\,ax^2 + 5\,a^3$.

   (a) Find the slope for $x = 0$ and $x = 2\,a$.
   (b) Find the points where the curve is parallel to the X-axis.
   (c) Find the points at which the slope of the curve is 1.

2. Find the angle at which the circle $x^2 + y^2 = 8\,x$ intersects the curve $y^2 = \dfrac{x^3}{2 - x}$.

3. Investigate the following curves by means of their derivatives:

$$\text{(a) } y = x^2(a - x)^2. \qquad \text{(b) } y = \frac{1}{1 + x^2}. \qquad \text{(c) } y = \frac{x(x - 2)}{2\,x - 5}.$$

4. (a) Find the equations of the tangent and normal to the curves of Ex. 3 at the following points: $x = 2a$; $x = 1$; $x = 5$.

(b) At the same points find the lengths of the tangent and normal; also the subtangent and subnormal.

5. In the case of the semi-cubical parabola $ay^2 = x^3$ show that

$$\text{subnormal} = (\text{subtangent})^2 \cdot \frac{27}{8a}.$$

6. (a) Find the equation of a curve whose polar subnormal is constant. (b) Find a curve whose polar subtangent is constant.

7. Test the following equations for multiple roots:

(a) $x^4 - 9x^2 + 4x + 12 = 0.$
(b) $x^5 - 6x^3 + 3x^2 - 7x - 18 = 0.$
(c) $x^4 - 6x^3 - 3x^2 + 56x - 48 = 0.$

8. Show that Rolle's theorem does not hold for $f(x) = (x - 1)^{\frac{2}{3}} - 1$ between $x = 0$ and $x = 2$. Explain why.

9. Apply Eq. (5) Art. 32 to the function $f(x) = 2x^2 + 5x - 8$. Find the value of θ for $a = 4$, $h = 1$; also for $a = 7$, $h = 2$.

10. Show that for every quadratic function $f(x) = ax^2 + bx + c$, Eq. (5), Art. 32 is satisfied when $\theta = \frac{1}{2}$. From this fact deduce a geometric property of the parabola.

11. If Q denotes quantity of heat, τ temperature, s length or distance, and t time, express in words what is meant by the equation

$$D_t Q = -k \cdot D_s \tau,$$

which applies to the flow of heat along a bar.

12. If v_1 and v_2 are the volumes corresponding to temperatures τ_1 and τ_2, respectively, show that approximately

$$\frac{v_2}{v_1} = C_3(\tau_2 - \tau_1) + 1,$$

where C_3 is the coefficient of cubical expansion.

13. Show that the coefficients of superficial and cubical expansion are respectively two and three times the coefficient of linear expansion, very nearly.

14. Find the volume elasticity of a gas which during isothermal expansion follows van der Waals' equation

$$\left(p + \frac{a}{v^2}\right)(v - b) = C.$$

15. Let m denote the mass of a body moving in a straight line, v the speed, and a the acceleration of the body, F the constant force acting on it, and s the distance traversed. From mechanics we have the following definitions and symbols:

product mv = momentum of body, (M).
product ma = force acting, (F),
product Fs = work of force F, (W).

Show that $F = D_t(mv) = D_t M,$

and also that $F = D_s W.$

16. The kinetic energy of the body (Ex. 15) is given by the expression

$$T = \tfrac{1}{2} mv^2.$$

(a) Show that the momentum is the v-derivative of the kinetic energy. (b) Show that the time-derivative of the kinetic energy is Fv.

17. Take $m = 2$, $F = 3$, and with assumed values of t as abscissas, draw curves for v, F, M, W, and T. Study the relations derived in Ex. 15 and 16 in connection with these curves.

18. The equation of Clausius

$$p = \frac{RT}{(v - a)} - \frac{C}{T(v + b)^2}$$

is sometimes used instead of van der Waals' equation. With T fixed find $D_v p$, and by means of this derivative investigate the general form of the isothermal curves. (See Ex. 2, Art. 24.)

19. Investigate both Clausius' and van der Waals' equations (with T constant) for points of discontinuity.

20. The following values of corresponding pressures and temperatures of saturated steam are taken from a standard table:

p	lb./sq. in.	84	85	86	87	88
τ	temperature F.	315.19°	316.02°	316.84°	317.65°	318.45°.

Making use of the law of the mean, find approximately the value of the derivative $D_p\tau$ for $p = 85$; also for $p = 87$.

21. A curve having the equation $\tau = ap^{\frac{1}{4}} + b$ will fit very closely the points given by the values of p and τ, Ex. 20. Determine the constants from the pressures 84 and 86, then from the equation determine $D_p\tau$ for $p = 85$ and compare the result with that found previously.

CHAPTER V

ANTI-DERIVATIVES OF SIMPLE ALGEBRAIC FUNCTIONS

38. Definition of anti-derivative. Thus far we have been concerned in finding the derivatives of given functions. We shall now consider the inverse operation; that is, having given a function $\phi(x)$, to find another function $f(x)$ such that $D_xf(x) = \phi(x)$. This inverse operation is called *integration,* and the resulting function is called the *integral or anti-derivative* of the given function. The function integrated is called the *integrand.* We denote the operation of integration by prefixing the symbol D_x^{-1} to the integrand. $D_x^{-1}\,\phi(x)$ is read "anti-derivative of $\phi(x)$."

Ex. Find the anti-derivative of the function $\phi(x) = x$. We have by inspection

$$D_x^{-1}x = \tfrac{1}{2}\,x^2.$$

This can be verified by actually differentiating the function $\frac{1}{2}\,x^2$, thus obtaining the given function x.

It is to be noted that the same derivative x would have been obtained, had we differentiated the function $\frac{1}{2}\,x^2 + C$, where C is any constant. The general form of the anti-derivative is thus $\frac{1}{2}\,x^2 + C$. Since the derivative of a function plus a constant is always equal to the derivative of the function, the general form of the anti-derivative or integral of a function must involve an additive constant. This constant is an arbitrary constant, and is called the *constant of integration.* Its value, however, in any particular case may be determined when certain initial conditions in the problem under discussion are known. The determination of this constant from such conditions will be more fully discussed in the next article.

An anti-derivative or integral function may be represented by a graph. The geometrical significance of annexing the constant of integration is to increase or decrease each ordinate of this graph by the same length. In other words, as we give various values to the constant of integration, we move the integral curve up or down. In the example just considered, the integral curves are parabolas of the same form as the parabola $y = \frac{1}{2} x^2$. This system of curves is shown in Fig. 28.

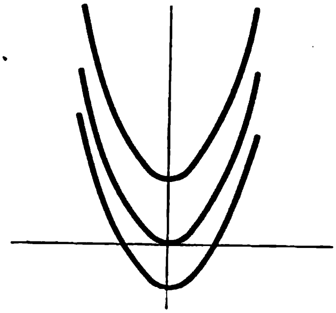

Fig. 28.

39. Two general theorems. It has been shown that the derivative of the product of a constant and a function is the product of the constant and the derivative of the function; that is, $D_x C\cdot f(x) = C\cdot D_x f(x)$. The inverse operation gives a similar theorem, which is expressed by the following equation:

$$D_x^{-1}\, C\quad f(x) = C \cdot D_x^{-1} f(x). \tag{1}$$

THEOREM I. *A constant factor in the integrand may be written either before or after the symbol D^{-1}.*

The derivative of the sum of a finite number of functions was obtained by adding the derivatives of the separate functions. The inverse operation gives, therefore, the following theorem:

THEOREM II. *The integral of the algebraic sum of a finite number of functions is equal to the algebraic sum of the integrals of these functions; that is,*

$$D_x^{-1}(u_1 + u_2 + \ldots + u_n) = D_x^{-1} u_1 + D_x^{-1} u_2 + \ldots + D_x^{-1} u_n. \tag{2}$$

40. Integrals of x^n and u^n. If the integrand is $\phi(x) = x^n$, where n has any value except -1, we have

$$D_x^{-1} x^n = \frac{x^{n+1}}{n+1} + C,\ (n \neq -1), \tag{1}$$

as may be verified by differentiation.

Ex. 1. $D_x^{-1} x^5 = \frac{1}{6} x^6 + C.$

Ex. 2. $D_x^{-1}\, 3x^2 = 3 D_x^{-1} x^2 = 3\left(\frac{1}{3} x^3 + C\right) = x^3 + C'.$

If $u = f(x)$, *we have the following more general formula:*

$$D_x^{-1}(u^n D_x u) = \frac{u^{n+1}}{n+1} + C,\ (n \neq -1). \tag{2}$$

Ex. 3. Find $D_x^{-1}\, x\,(3x^2 + 7)^4$.

Put $u = 3x^2 + 7$. Then $D_x u = 6x$, and the integrand takes the form

$$\tfrac{1}{6}(3x^2 + 7)^4\, 6x = \tfrac{1}{6} u^4 D_x u.$$

$$D_x^{-1}\left(\tfrac{1}{6} u^4 D_x u\right) = \tfrac{1}{6} D_x^{-1}\left(u^4 D_x u\right) = \tfrac{1}{6}\left(\tfrac{1}{5} u^5 + C\right) = \tfrac{1}{30}(3x^2 + 7)^5 + C'.$$

Ex. 4. Find the anti-derivative of $\dfrac{3x-2}{\sqrt{3x^2-4x-2}}$.

If we put $u = 3x^2 - 4x - 2$, we have $D_xu = 6x - 4 = 2(3x - 2)$, and the integrand becomes $\frac{1}{2}u^{-\frac{1}{2}} \cdot D_xu$. Then

$$D_x^{-1}(\tfrac{1}{2}u^{-\frac{1}{2}} \cdot D_xu) = u^{\frac{1}{2}} + C = \sqrt{3x^2 - 4x - 2} + C.$$

EXERCISES.

Verify the following:

1. $D_x^{-1}x^5 = \frac{1}{6}x^6 + C.$
2. $D_x^{-1}(3x^2 + x - 5) = x^3 + \frac{1}{2}x^2 - 5x + C.$
3. $D_x^{-1}(ax^3 + bx^2 + cx + d) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx + C.$
4. $D_x^{-1}(3x^{-2} - 5x^{-4}) = -3x^{-1} + \frac{5}{3}x^{-3} + C.$

Find the anti-derivatives of the following functions:

5. $ax^{-3} + bx^{-2}.$
6. $5x^6 - 4x^3 + 2x + 7.$
7. $3x^{-2} + 4 - x^2 + 2x^4.$
8. $\dfrac{1}{x^3} + \dfrac{1}{x^4} + \dfrac{1}{x^6}.$
9. $x^{\frac{3}{2}} - 3x^{\frac{1}{2}}.$
10. $x^{-\frac{3}{2}} - 2x^{-\frac{1}{2}}.$
11. $\dfrac{x^2 - 4x^3}{x^5}.$
12. $(x - 1)^2.$
13. $(2x^2 - 3x)^4.$
14. $(ax + b)^2.$
15. $3x^2(x^3 + 4)^3.$
16. $x^3(2x^4 - 5)^3.$
17. $x(3x^2 - 7)^4.$
18. $\dfrac{3x^2 - 5}{\sqrt{x^3 - 5x + 7}}.$
19. $\dfrac{x - 3}{\sqrt[3]{x^2 - 6x + 1}}.$
20. $\dfrac{5mx}{\sqrt{1 + k^2x^2}}.$
21. Remembering that integration is the operation inverse to differentiation, formulate from the laws of differentiation all the laws you can for integration.

41. Integral curves. From a given function

$$y = f(x) \tag{1}$$

we obtain an integral function

$$y = D_x^{-1}f(x) = f_1(x) + C. \tag{2}$$

The curve whose equation is (2) is called the *first integral curve* of the curve whose equation is (1). Evidently also curve (1) is the derived curve of (2). If now we take the anti-derivative of (2), we get a third function,

$$y = D_x^{-1}[f_1(x) + C] = f_2(x) + C', \tag{3}$$

the graph of which is the *second integral curve* of curve (1).

As an example, we may take the equation for a falling body, Art. 34, Ex. 2. If the body falls from rest, the constants of integration are zero (see Art. 43), and we have

$$a = g,$$

and $$v = gt.$$

The original curve representing the equation $a = g$ is a straight line AB, Fig. 29, parallel to the t-axis. The first integral curve represents the integral function $v = gt$, and is evidently the straight line OC passing through the origin. The ordinates of this curve represent velocities. Taking the antiderivative of the function $v = gt$, we get

$$D_t^{-1}(gt) = \tfrac{1}{2}gt^2 = s, \qquad (3)$$

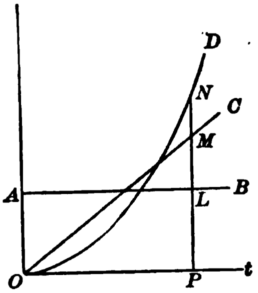

Fig. 29.

the graph of which is the parabola OD. Then OD is the first integral curve of curve OC and the second integral curve of curve AB. The ordinates of OD represent distances.

It will be observed that in this example the ordinates of the different curves represent magnitudes of different kinds, acceleration, velocity, and length. In the important applications of integral curves to mechanics, this is usually the case.

Let any point P be taken on the t-axis, and let the ordinate through P cut the curves in L, M, and N, respectively. Let t_1 denote the time represented by the abscissa OP; then the velocity represented by the ordinate PM is

$$v_1 = gt_1.$$

But since $OA = g$ and $OP = t_1$, the product gt_1 is also represented by the area $OALP$. Hence, for a given abscissa $OP = t_1$, the area under the original curve AB and the ordinate of the first integral curve represent the same thing, viz., the velocity v_1. The ordinate PN represents the distance

$$s_1 = \tfrac{1}{2}gt_1^2.$$

But since $PM = v_1 = gt_1$ and $OP = t_1$, the triangular area OPM, also represents $s_1 = \frac{1}{2}gt_1^2$. Again, the ordinate OD of the integral curve has the same numerical value as the area under the original curve OC. In this example, at least, the following statement holds good: For the same abscissa the same number indicates the length of an ordinate of the first integral curve and the area between the original curve, the axes, and the ordinate corresponding to the abscissa. In a subsequent chapter we shall deduce a general law covering this relation.

42. Curves having given properties. It has been shown that the value of the derivative of a function $y = f(x)$ for a particular value of the variable gives the slope of the tangent to the curve representing the function for the value of the variable. Furthermore, it has been shown that the derivative enters into the equation of the tangent and normal to the curve and into the expressions for the length of tangent, length of normal, subtangent, and subnormal. We have so far considered direct problems in which from a given function the derivative was found, and from this derivative the desired property of the curve was obtained. We shall now consider inverse problems in which the property of the curve is given, and from it the equation of the curve is to be obtained. These problems naturally involve the determination of an anti-derivative.

Ex. **1.** Determine the equation of a curve at every point of which the slope is equal numerically to one half the abscissa.

We have here

$$D_x y = \tfrac{1}{2} x,$$

whence

$$y = D_x^{-1} \tfrac{1}{2} x,$$

and taking the anti-derivative, we obtain

$$y = \tfrac{1}{4} x^2 + C.$$

This is the equation of a parabola having the Y-axis as the axis of the curve. By giving the constant C different values, we get a series of parabolas having the same slope for the same value of x. (See Fig. 28.)

Ex. **2.** Find the equation of the curve whose polar subtangent has a constant length a.

In this case, we have

$$\rho^2 D_\rho \theta = a,$$

whence

$$D_\rho \theta = \frac{a}{\rho^2} = a\rho^{-2},$$

or

$$\theta = D_\rho^{-1} a\rho^{-2}.$$

Performing the required operations, we obtain

$$\theta = -\frac{a}{\rho} + C$$

or

$$\rho = \frac{a}{C - \theta}$$

EXERCISES.

1. Find the equations of the curves whose slopes are respectively:

$$3; \quad \tfrac{1}{2} x^2; \quad mx; \quad -\frac{1}{x^2}; \quad ax - b; \quad \frac{1}{ax^3}.$$

2. Find the equation of the curve whose slope at any point is double the abscissa at that point.

3. Find the equation of a curve whose subnormal is b times the ordinate.

4. Find the equation of the curve whose subnormal varies inversely as the ordinate.

5. Find the polar equation of the curve for which $\tan \psi = k\rho$.

6. Find the polar equation of the curve whose polar subnormal has a constant length m.

7. Find the equation of the curve which passes through the point (2, 3) and whose tangent has the slope $3x + 5$.

43. Rectilinear motion. The relations between the time, speed, acceleration, and space traversed by a point moving in a straight line are:

$$v = D_t s, \quad a = D_t v.$$

We have, therefore, the inverse relations

$$v = D_t^{-1} a, \tag{1}$$

$$s = D_t^{-1} v. \tag{2}$$

In the direct problem, starting with the space traversed as a function of the time, we were able, by taking derivatives, to find the speed and acceleration. In the inverse problem, having the acceleration given as a function of the time, we can by finding anti-derivatives determine first the speed and second the space traversed.

Ex. 1. A point moves in a straight line with a constant acceleration a. The speed and the space passed over at the end of a given time-interval are required.

We have

$$v = D_t^{-1} a,$$

whence

$$v = at + C. \tag{3}$$

To determine the constant C, let v_0 denote the initial speed, that is, the speed when $t = 0$. Substituting these values in (3), we have

$$v_0 = 0 + C, \text{ or } C = v_0;$$

hence (3) becomes

$$v = at + v_0. \tag{4}$$

We have further

$$s = D_t^{-1} v = D_t^{-1} (at + v_0),$$

whence, performing the indicated operation, we obtain

$$s = \tfrac{1}{2} at^2 + v_0 t + C'. \tag{5}$$

To determine the constant C', let s_0 denote the initial space, that is, the space when $t = 0$. Substituting in (5)

$$s_0 = 0 + C', \text{ or } C' = s_0,$$

and we have finally

$$s = \tfrac{1}{2} at^2 + v_0 t + s_0. \tag{6}$$

An important application of these laws is to the case of freely falling bodies. By observation, it is found that a body falling freely in a vacuum at a given point on the earth's surface has a constant acceleration. This acceleration, while constant for any one place, varies for different localities within small limits. Its value may be taken as 32.2 ft./sec.^{2}.

If the body falls from rest, we have $v_0 = 0$ and $s_0 = 0$. Denoting by g the constant acceleration, we have, therefore, from (4) and (6),

$$v = gt, \tag{7}$$

$$s = \tfrac{1}{2} gt^2. \tag{8}$$

Eliminating t between (7) and (8), we get as a third relation,

$$v^2 = 2\,gs,$$

or

$$v = \sqrt{2\,gs}. \tag{9}$$

Ex. **2**. **Investigate the motion of a body projected vertically upward from the earth's surface with an initial velocity of b feet per second.**

We have here $v_0 = b$ and $s_0 = 0$. The acceleration is g, but in this case is opposite to the direction of motion; hence $a = -g$. From the fundamental relations (1) and (2), we have upon integration,

$$v = -gt + b,$$
$$s = -\tfrac{1}{2} gt^2 + bt.$$

The body will reach its highest point when $v = 0$, that is, when

$$0 = -gt + b, \text{ or } t = \frac{b}{g},$$

and the height h to which it rises is found by substituting this value of t in the second equation. Thus

$$h = -\tfrac{1}{2}\,g\left(\frac{b}{g}\right)^2 + b\cdot\frac{b}{g} = \frac{b^2}{2g}.$$

44. Rotation about a fixed axis. Corresponding to the direct relations

$$\omega = D_t\theta, \qquad \alpha = D_t\omega,$$

we have the inverse relations

$$\omega = D_t^{-1}\alpha, \tag{1}$$

$$\theta = D_t^{-1}\omega, \tag{2}$$

and starting with the angular acceleration α as a given function of the time, we can by finding anti-derivatives determine the angular speed and the angle swept through by a given radius. For the case in which the angular acceleration is a constant, say α_0, we readily derive the following formulas for ω and θ:

$$\omega = \alpha_0 t + \omega_0. \tag{3}$$

$$\theta = \tfrac{1}{2}\alpha_0 t^2 + \omega_0 t + \theta_0. \tag{4}$$

Ex. A hoisting drum has an angular speed of 30 rad./sec. To bring it to rest a brake is applied in such a way as to produce a retardation (i.e. negative angular acceleration) of 3 rad./sec.². How many turns will the drum make before coming to rest?

We have here $\omega_0 = 30$ and $\alpha_0 = -3$. The drum will come to rest when $\omega = 0$, that is, when

$$\alpha_0 t + \omega_0 = 0.$$

Hence
$$t = -\frac{\omega_0}{\alpha_0} = \frac{30}{3} = 10 \text{ sec.}$$

To find the angle turned through after the brake is applied, we must take $\theta_0 = 0$, and substitute 10 for t in equation (4). Thus

$$\theta = \tfrac{1}{2}\alpha_0 t^2 + \omega_0 t = -\tfrac{3}{2} \times 10^2 + 30 \times 10 = 150 \text{ radians.}$$

Since 1 revolution is equal to 2π radians, the number of revolutions is

$$\frac{150}{2\pi} = 23.86.$$

EXERCISES.

1. A point moves in a straight line in such a way that the acceleration is given by the equation $a = 20 - 3t$. (a) Starting from rest, what will be the speed at the end of 13 seconds? (b) How far will the point have moved in 8 seconds?

2. Let the speed of a point moving in a straight line be given by the equation $v = t^2 - 6t$. Derive expressions for the acceleration and for the distance traversed in t seconds, starting from rest.

3. A ball is thrown vertically upward with an initial speed of 225 ft./sec. Find the height to which it will rise and the time that will elapse before it reaches the earth.

4. A disk rotates about a fixed axis with an angular acceleration given by the equation $\alpha = 10 - t$.

(a) Find the angular speed at the end of 5 seconds, if $\omega_0 = 30$.

(b) Find the number of revolutions the disk will have made in 5 seconds.

(c) Find the number of revolutions the disk will make before coming to rest.

MISCELLANEOUS EXERCISES.

1. Find integrals of the following functions:

(a) $(3x - 5)^4$. (b) $x^2(x^3 - 4)^2$. (c) $x^2\sqrt{x^3 - 4}$. $\frac{2}{9}\sqrt{\ }$
(d) $x\sqrt[3]{3x^2 - 5}$. $\frac{1}{8}\sqrt[3]{\ }$ [illegible]

2. Find the equation of a curve whose tangent has the slope $4 - x^2$ and which passes through the point $(3, -2)$. $y = -\frac{x^3}{3} + 4x - 5$

3. Find the equation of a curve whose subnormal has a constant length a

4. Determine curves whose slopes are respectively

(a) $5x - 4$. (b) $3x^2 - x + 2$. (c) $x^{\frac{1}{3}}$. (d) $\sqrt{x} + 2$.

Assume a point on each curve, and thus determine the constant of integration in each case.

5. In the theory of the bending of beams, x-anti-derivatives of the following functions are required. Find these anti-derivatives and determine the constants of integration from the conditions given.

(a) $f(x) = \frac{1}{2}w(L - x)^2$, $D^{-1}f(x) = 0$, when $x = 0$.
(b) $f(x) = L^2x - Lx^2 + \frac{1}{3}x^3$, $D^{-1}f(x) = 0$, when $x = 0$.
(c) $f(x) = \frac{1}{8}wL^2x - \frac{1}{6}wx^3$, $D^{-1}f(x) = 0$, when $x = 0$.
(d) $f(x) = \frac{3}{8}wLx - \frac{1}{2}wx^2$, $D^{-1}f(x) = 0$, when $x = L$.

6. Find expressions for the acceleration and distance traversed when the speed is given by the following equations, respectively:

(a) $v = at - \frac{b}{t^2}$. (b) $v = c\sqrt{t}$. (c) $v = t^2 - 5t + 4$.

7. Since the specific heat $C = D_\tau Q$, then inversely $Q = D_\tau^{-1}C$. From the equation $C = a + b\tau + c\tau^2$ derive an expression for Q.

8. Derive inverse relations from the direct relations given in Ex. 15, 16, pp. 84 and 85.

CHAPTER VI

THE DIFFERENTIAL NOTATION

45. The derivative as a rate. The fundamental problem of differential calculus is the measurement of the rate of change of the function with respect to the variable. We are not concerned so much with the actual change of the function as with its change per unit increase of the variable, or, in other words, its *rate* of change. The great importance of the derivative lies in the fact that it gives a precise measure of this rate of change. Thus the derivative D_ts measures the rate of change of the distance s with respect to the time, or, more briefly stated, the time-rate of s; the derivative $D_t\theta$, the angular speed, is likewise the time-rate of the angle θ. Again, consider the case of heating a metal bar. The length x of the bar is a function of its temperature τ, say $x = f(\tau)$, and the derivative $D_\tau x$ gives the rate of change of length with respect to the temperature.

That the derivative measures the rate of change of the function is easily seen. If Δy denotes the change of the function corresponding to a change Δx of the variable, the quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

gives the average rate of change for the interval Δx. The rate at which y is changing with respect to x at any particular value of x, as x_0, is the limiting value of this quotient as Δx approaches zero, that is,

$$\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = f'(x_0).$$

It is not necessary that y be expressed directly in terms of x in order that we may discuss their comparative rates of change. In fact, it is often convenient to compare the rates of change of two functions by means of a third variable. For example, we may compare the relative rate of change in the market value of wheat in Chicago and steel rails in New York by the use of a third variable representing money. The speed of a locomotive may be compared with that of a street-car, or the relative speed of two chemical reactions may be found by means of a third variable representing time. Again, we may compare two electromotive forces by the currents they will develop in the same circuit.

By considering x and y as functions of a third variable, we are led to a new method of expressing rates of change, or derivatives, namely, by means of the differential notation.

46. Differential notation. If x and y are regarded as functions of a third variable t, say

$$y = \phi(t), \quad x = \psi(t),$$

y is a function of x, since by eliminating t, y can be expressed directly in terms of x. We may therefore write

$$y = f(x), \quad x = \psi(t).$$

Hence by Art. 21, we have

$$D_t y = f'(x)\, D_t x. \qquad (1)$$

It is customary to write this expression in a simpler form as follows:

$$dy = f'(x)\, dx. \qquad (2)$$

The symbols dy and dx are called *differentials*.* As the differentials enter homogeneously into the equation, it is not necessary that dy be equal to $D_t y$, and dx to $D_t x$; it is sufficient that they be proportional to these derivatives.

The differentials dx and dy should be carefully distinguished from the increments Δx, Δy (see Art. 15). This distinction may be clearly shown by a geometrical illustration. Let $y = f(x)$ be a continuous function represented by the curve shown in Fig. 30. Draw the tangent at any point P of this curve and let x take an increment Δx equal to PS. Δx has a certain arbitrary ratio to the t-rate of change of x, that is to $D_t x$, as the

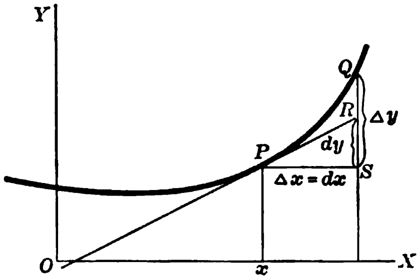

Fig. 30.

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* Some authors define differentials differently. See Byerly's *Differential Calculus*, Art. 178; Gibson's *Calculus*, Art. 60; Murray's *Calculus*, Art. 27.

moving point passes through P. Because of the arbitrary character of Δx, we may take this ratio to be the same as the ratio which dx bears to $D_t x$, and consequently we have $dx = \Delta x$. However, the equality of Δy and dy does not usually hold. dy, equal to SR in the figure, is the product of $f'(x)$ and Δx; that is, it measures the change which the function would make if the rate at P were maintained throughout the interval Δx. On the other hand, Δy, given by SQ in the figure, is the change which the function makes if the moving point is restricted to the given curve, and is measured, therefore, from S to the point Q on the curve rather than to R on the tangent. Δy is identical with dy only in the special case in which the rate of change of y with respect to x is constant in the interval Δx.

From equations (1) and (2) we have at once

$$D_x y = f'(x) = \frac{dy}{dx}, \tag{3}$$

that is, *a derivative is the quotient of two differentials.* It is frequently convenient to write the derivative as such a quotient, and in the future we shall employ either $D_x y$ or $\frac{dy}{dx}$ as best suits the problem under discussion.

We may read $\frac{dy}{dx}$ either "the derivative of y with respect to x," or "differential y divided by differential x."

The distinction between the symbols $D_x y$ and ~~$\frac{dx}{dy}$~~ should be $\frac{dy}{dx}$ carefully noted. While both lead to the same numerical result and may therefore be used interchangeably, $D_x y$ indicates that a certain operation has been performed upon the function y with respect to the independent variable x. It is not possible to separate this symbol into two parts. On the other hand, $\frac{dy}{dx}$ is merely a quotient, which can be dealt with by the ordinary rules of algebra.

Because of the form of equation (2), $f'(x)$ is sometimes called the *differential coefficient,* that is, the coefficient of the differential of the independent variable.

47. Kinematic illustration of differentials. An instructive illustration of the use of differentials is afforded by the velocity components of a moving point.

Suppose a point P to move along a curve m, Fig. 31, and let P_x and P_y be the projections of P on the X-axis and Y-axis respectively. As P moves on the curve, the projections P_x and P_y will move on the axes. The velocity of P_x along OX is evidently the time-rate of change of the abscissa x, and is therefore given by the derivative D_tx. Likewise, the velocity of P_y along the Y-axis is D_ty. The velocity of P along the curve has the direction of the tangent PT, and its magnitude is given by the derivative D_ts. Suppose that the velocity of P is such that in a unit of time P would be carried along the tangent from P to T. This displacement PT can be effected in the following manner: Move P along a horizontal line from P to A, and at the same time move the horizontal line PA vertically to BT. Evidently, if the horizontal and vertical motions are made with constant velocity, the point P will move along PT with constant velocity. Since the motion takes place in a unit of time the displacements PT, PA, and PB represent respectively the velocity in the path, the velocity of the horizontal motion of P_x, and the velocity of the vertical motion of P_y; that is, $PT = D_ts$, $PA = D_tx$, and $PB = D_ty$.

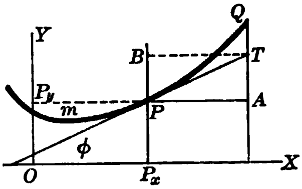

Fig. 31.

The two velocities D_tx and D_ty, which together may replace the velocity D_ts in the curve, are called the *component velocities* in the direction of the axes. From the figure the following relations between the velocity in the curve and the components along the axes are evident.

$$\overline{PT}^2 = \overline{PA}^2 + \overline{PB}^2,$$

hence

$$(D_ts)^2 = (D_tx)^2 + (D_ty)^2.$$

Substituting the differentials ds, dx, and dy for the time-rates, we have

$$ds^2 = dx^2 + dy^2. \qquad (1)$$

Furthermore,

$$PA = PT \cos \phi,$$

and

$$PB = PT \sin \phi;$$

that is,

$$\left.\begin{aligned} dx &= ds \cos \phi \\ dy &= ds \sin \phi \end{aligned}\right\}. \qquad (2)$$

Ex. 1. A point moves along the straight line $5x - 12y = 7$, in such a way that the X-component of its velocity is 3 ft./sec. Find the Y-component and the velocity of the point in the given line.

From the given equation, $5x - 12y = 7$, we have by differentiation

$$5\,D_t x - 12\,D_t y = 0,$$

or

$$5\,dx - 12\,dy = 0.$$

Therefore,

$$dy = \tfrac{5}{12}\,dx = 1\tfrac{1}{4}\text{ ft./sec.}$$

The velocity in the given straight line is from (1)

$$ds = \sqrt{3^2 + (\tfrac{5}{4})^2} = 3\tfrac{1}{4}\text{ ft./sec.}$$

Ex. 2. A point moves in the parabola $y^2 = 8x$ with a constant velocity of 5 units. Find the components along the axes when the point is at (12.5, 10).

From the given equation, we get

$$2y\,D_t y = 8\,D_t x,$$

or

$$2y\,dy = 8\,dx,$$

whence

$$dy = \frac{4}{y}\,dx.$$

But

$$ds = \sqrt{dx^2 + dy^2} = 5,$$

whence

$$dx\sqrt{\frac{16}{y^2} + 1} = 5.$$

We have then

$$dx = \frac{5y}{\sqrt{16 + y^2}},$$

which becomes for $y = 10$

$$dx = \frac{50}{\sqrt{116}} = \frac{25}{29}\sqrt{29};$$

and

$$dy = \frac{4}{10}\cdot\frac{25}{29}\sqrt{29} = \frac{10}{29}\sqrt{29}.$$

In case a point moves along a curve whose equation is given in polar coördinates, it is convenient to resolve the velocity along the curve into components along and perpendicular to the radius vector, respectively. Thus in Fig. 32, the velocity represented by PT may be resolved into components represented by PA and PB. As before, $PT = D_t s$. The velocity component PA is evidently the time-rate of increase of the radius vector $OP = \rho$, that is

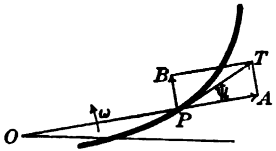

Fig. 32.

$$PA = D_t\rho.$$

The component PB is the velocity that P would have if it remained at rest on the radius while the latter rotated about the pole O; hence if ω denotes the angular speed of OP about the pole O,

$$PB = \overline{OP}\,\omega = \rho D_t\theta. \qquad \text{(See Art. 36.)}$$

Since

$$\overline{PT}^2 = \overline{PA}^2 + \overline{PB}^2,$$

$$(D_ts)^2 = (D_t\rho)^2 + (\rho D_t\theta)^2,$$

or, substituting differentials for the time-derivatives,

$$ds^2 = d\rho^2 + \rho^2 d\theta^2. \tag{3}$$

It follows also from the figure that

$$\left.\begin{aligned} \tan\psi &= \frac{\rho d\theta}{d\rho} \\ d\rho &= ds\cos\psi \\ \rho d\theta &= ds\sin\psi \end{aligned}\right\} \tag{4}$$

Ex. 3. A point moves in the curve $\rho = a\theta$ with a constant velocity m. What are the velocity components when $\theta = \pi$?

Since $\rho = a\theta$,

we have $D_t\rho = aD_t\theta$,

or $d\rho = a\,d\theta$.

Then

$$d\rho^2 = a^2d\theta^2,$$

$$ds^2 = d\rho^2 + \rho^2d\theta^2 = (a^2 + \rho^2)\,d\theta^2,$$

$$ds = m = (a^2 + \rho^2)^{\frac{1}{2}}\,d\theta,$$

or

$$d\theta = \frac{m}{\sqrt{a^2 + \rho^2}},$$

whence

$$\rho d\theta = \frac{m\rho}{\sqrt{a^2 + \rho^2}},$$

and

$$d\rho = \frac{ma}{\sqrt{a^2 + \rho^2}}.$$

For $\theta = \pi$, $\rho = a\pi$. Substituting, we obtain

$$\rho d\theta = \frac{m\pi}{\sqrt{1 + \pi^2}}, \text{ velocity perpendicular to } OP,$$

$$d\rho = \frac{m}{\sqrt{1 + \pi^2}}, \text{ velocity along } OP.$$

48. Differentiation with differentials. The passage from the derivative to the differential notation, or vice versa, is effected by the defining equation

$$dy = f'(x)\,dx. \tag{1}$$

Thus, if

$$y = a\,x^2 + b,$$

$$f'(x) = 2\,ax,$$

whence

$$dy = 2\,ax\,dx.$$

Conversely, if we have given

$$dy = (3\,x^2 + 4)\,dx,$$

we obtain by division

$$\frac{dy}{dx} = f'(x) = 3\,x^2 + 4.$$

In the operation of differentiation, we may employ either of the following methods:

(1) We may obtain the derivative directly by the theorems already established; or,

(2) We may derive an equation in the form (1) which is homogeneous in the differentials of the variables, and then obtain the derivative by division. The latter method is frequently the more convenient and expeditious.

The general theorems of Art. 19 may be easily restated in terms of the differential notation. All that is necessary is to replace D_xy, D_xu, D_xv, etc., by their equivalents $\frac{dy}{dx}$, $\frac{du}{dx}$, $\frac{dv}{dx}$, etc., and multiply by dx. The following are the results thus obtained, which the student may easily verify.

(a) Given $y = c$, $dy = 0$.
(b) Given $y = x$, $dy = dx$.
(c) Given $y = (u + v + w)$, $dy = du + dv + dw$.
(d) Given $y = u + c$, $dy = du$.
(e) Given $y = uv$, $dy = udv + vdu$.
(f) Given $y = cu$, $dy = cdu$.
(g) Given $y = \dfrac{u}{v}$, $dy = \dfrac{vdu - udv}{v^2}$.

The following examples will illustrate the use of the preceding formulas:

Ex. 1.

$$\begin{aligned} y &= 5x^3 - 4x^2 + x - 10, \\ dy &= 15x^2dx - 8x\,dx + dx \\ &= (15x^2 - 8x + 1)\,dx. \end{aligned}$$

Ex. 2.

$$\begin{aligned} y &= x(x^2 - a^2), \\ dy &= (x^2 - a^2)\,dx + x \cdot d(x^2 - a^2) \\ &= (x^2 - a^2)\,dx + 2x^2dx \\ &= (3x^2 - a^2)\,dx. \end{aligned}$$

Ex. 3.

$$y = \frac{x}{\sqrt{x^2 - a^2}}\,.$$

$$dy = \frac{\sqrt{x^2 - a^2}\,dx - x \cdot d\sqrt{x^2 - a^2}}{(\sqrt{x^2 - a^2})^2}$$

$$= \frac{\sqrt{x^2 - a^2}\,dx - \dfrac{x^2dx}{\sqrt{x^2 - a^2}}}{x^2 - a^2} = -\frac{a^2dx}{\sqrt{(x^2 - a^2)^3}}\,.$$

EXERCISES

Differentiate the following functions, and express the results in differential form:

1. $y = x^5.$
2. $y = 3x^2 + 7.$
3. $y = (3x - 5)(x^2 + 2).$
4. $y = x(x - 1)(x - 2)(x - 3).$
5. $y = -\dfrac{1}{x^5} + 7.$
6. $y = \dfrac{x - 3}{x - 5}\,.$
7. $y = \dfrac{x\sqrt{a^2 - x^2}}{\sqrt{a^2 + x^2}}\,.$
8. $y = x^{\frac{1}{2}}(x - a)^{-\frac{1}{2}}.$
9. $y = x^6 - 3x^4 + 5x^2 - 12.$
10. $y = \sqrt[3]{3x^3 + 4x - 8}.$
11. $r = \frac{1}{5}\theta^3 - \frac{1}{3}.$
12. $s = at^2 + bt + c.$
13. $Q = a + b\tau + c\tau^2.$
14. $u = B + \dfrac{C}{T} + \dfrac{D}{T^2}\,.$
15. $p = \dfrac{c}{v^m}\,.$
16. $y = (a + bx^n)^m.$

49. Differentiation of implicit functions. It frequently happens that the relation between two variables, as x and y, is conveniently expressed by means of an implicit function. In such cases the derivative $f'(x)$ can be found by differentiating according to the general formulas just given and taking the quotient $\dfrac{dy}{dx}$. Another method will be discussed in Chapter XII. The expression for the derivative will generally contain both variables, and its numerical value can be found for known simultaneous values of x and y.

Thus, if the given equation is

$$x^3y - 4\,xy^2 + 6\,x^2 - 3\,y = 0,$$

we obtain by differentiation (using the formula for the product)

$$(3\,x^2y\,dx + x^3dy) - (4\,y^2\,dx + 8\,xy\,dy) + 12\,x\,dx - 3\,dy = 0,$$

whence

$$\frac{dy}{dx} = -\frac{3\,x^2y - 4\,y^2 + 12\,x}{x^3 - 8\,xy - 3}.$$

The values $x = 1, y = 1$, satisfy the original equation, and for these values the derivative has the value

$$\left.\frac{dy}{dx}\right]_{1,1} = \frac{11}{10} = 1.1;$$

that is, the slope of the curve is 1.1 at the point (1, 1).

Ex. 1. The equation representing the adiabatic expansion of air is $pv^k = C$, where k and C are constants. p denotes the pressure and v the volume of the air.

Differentiating, we obtain

$$v^kdp + kpv^{k-1}dv = 0,$$

or

$$v\,dp + kp\,dv = 0,$$

whence

$$\frac{dp}{dv} = -k\,\frac{p}{v}.$$

Ex. 2. Find the slope of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

We require an expression for $\frac{dy}{dx}$. From the equation of the ellipse, we have by differentiation,

$$\frac{2\,x\,dx}{a^2} + \frac{2\,y\,dy}{b^2} = 0,$$

whence

$$\frac{dy}{dx} = -\frac{b^2\,x}{a^2\,y}$$

Substituting the coördinates of the point, we get

$$\left[\frac{dy}{dx}\right]_{x_1\,y_1} = -\frac{b^2\,x_1}{a^2\,y_1}.$$

EXERCISES.

1. Find the derivative $\frac{dy}{dx}$ for each of the following functions:

(a) $x^2y - y^3 = 0.$

(b) $x^3y + x^2y^2 + 4\,xy^3 - y^4 = 0.$

(c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

(d) $xy + cy + ex + f = 0.$

(e) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

2. Find the general expression for the slope of the conic section whose equation is

$$ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0.$$

3. Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

4. Find the slope of the ellipse $9x^2 + 14y^2 = 50$ at the point $(2, 1)$; at the point $(-2, 1)$. $\mp \frac{9}{7}$

5. Find the slope of the curve

$$xy^3 - 3x^2y + 6y^2 + 2x = 0$$

at the point $(2, 1)$.

6. Derive expressions for the polar subtangent and polar subnormal of the curves:

$$\text{(a)}\quad \rho^2 - \rho\theta^{\frac{1}{2}} = C. \qquad\qquad \text{(b)}\quad \rho - \rho^2\theta + C = 0.$$

7. Derive a general expression for the derivative $\frac{dp}{dv}$ when the expansion of a gas follows the law

$$\text{(a)}\quad p^m v^n = C, \qquad (m, n, \text{ and } C \text{ constants});$$

$$\text{(b)}\quad \left(p + \frac{a}{v^2}\right)(v - b) = C.$$

8. Find $\frac{dy}{dx}$ in the following:

$$\text{(a)}\quad (1 + x)^2 y = (1 - x)y^2 - x^2. \qquad \text{(b)}\quad xy^2 = b(b - x).$$

$$\text{(c)}\quad x^5 - 3x^4y + x^2y^3 - 4y^5 = 0.$$

50. Applications of rates and differentials. Suppose we have given

$$y = f(x),$$

and by differentiation we obtain

$$dy = f'(x)\,dx.$$

By definition the differentials dy and dx are the derivatives $D_t y$ and $D_t x$ (or numbers proportional to them), where t denotes any third variable. If we consider the variable t as denoting time, then dy and dx are merely the time-rates of the function y and the variable x respectively. Hence, if the time-rate of x is given, the time-rate of y is found by differentiation.

Ex. **1.** The area A of a circular plate is a function of the radius r, the relation between them being

$$A = \pi r^2.$$

Suppose the plate to be heated, and let the radius be increasing uniformly at the rate of 0.1 inch per minute. At what rate is the area increasing when the radius is 6 inches?

Differentiating, we obtain

$$dA = 2\pi r\,dr.$$

By the given conditions, $dr = 0.1$ in./min., when $r = 6$ in. Hence

$$dA = 2\pi \times 6 \times 0.1 = 1.2\pi.$$

That is, when $r = 6$ in., the area is increasing at the rate of 1.2π sq. in. per minute.

Ex. 2. Boyle's law for the expansion of air is expressed by the equation

$$pv = C.$$

At a given instant the pressure is 40 lb./sq. in., the volume of the air is 8 cu. ft., and the volume is increasing at the rate of 0.5 cu. ft./sec. At what rate is the pressure changing?

We have $$pv = C,$$

whence $$p\,dv + v\,dp = 0,$$

or $$dp = -\frac{p}{v}\,dv.$$

$C = pV$

$p\,dv + v\,dp = 0$

Since now $dv = D_t v = 0.5$,

$$dp = -\frac{40}{8} \times 0.5 = -2.5.$$

Hence the pressure is *decreasing* at the rate of 2.5 lb./sq. in. per second.

Ex. 3. A moving point traces a parabola whose equation is $y^2 = 10x$. When $x = 4$, the abscissa is increasing at the rate of 2 units per second. What is the rate of increase of the ordinate?

$$y^2 = 10x,$$

$$2y\,dy = 10\,dx,$$

whence $$dy = \frac{10}{2y}\,dx.$$

For $x = 4$, $y = \sqrt{40}$, and $dx = 2$. Hence

$$dy = \frac{10}{2\sqrt{40}} \times 2 = \tfrac{1}{2}\sqrt{10}.$$

EXERCISES.

1. A point moves in the straight line $5x - 3y = 30$ in such a way that tht Y-component of its velocity is 8. Find the X-component and the velocity in the line.

2. Find the X- and Y-components of the velocity of a point which moves in the line $x + 2y = 12$ with a speed of 30 ft./sec.

3. Suppose that a straight wire rotates about one end with an angular speed of 3 rad./sec. and that a bead on this wire moves along it with a speed of 8 ft./sec. When the bead is 2 feet from the center of rotation, (a) what is the component of its velocity perpendicular to the wire? (b) What is its velocity in its path?

4. Find the polar equation of the curve described by the bead, Ex. 3, (a) when the angular speed ω of the wire is a times the linear speed of the bead along the wire; (b) when ω is constant and the speed of the bead on the wire varies inversely as the distance from the center of rotation.

5. If a soap bubble's diameter is increasing at the rate of $\frac{1}{4}$ in./sec. when it is 4 inches, at what rate is the inclosed volume increasing?

6. If a man 6 feet in height is walking at the rate of 3 mi./hr. away from a lamp-post 30 feet high, at what rate is the end of his shadow receding from the lamp-post?

7. The time of a complete oscillation of a pendulum of length L is given by the formula

$$t = 2\pi\sqrt{\frac{L}{g}}.$$

Find the rate of change of the time compared with that of the pendulum's length.

8. Find the equation of the path of a point which moves in such a way that the X-component of its velocity is constant while the Y-component varies directly as the time. Give a physical illustration.

9. The ends A and B of a line 20 inches in length move on two lines at right angles. If the velocity of B is 8 when 12 inches from O, find the velocity of A; also the velocity of the middle point of the line.

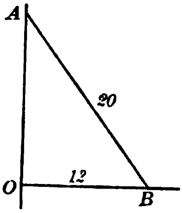

Fig. 33.

10. An engine cylinder has a diameter of 10 inches. At what speed is the piston moving when steam is entering the cylinder at the rate of 14 cu. ft./sec.?

51. Integrals written with differential notation. In Chapter V we deduced certain general laws for obtaining the anti-derivative or integral of a given function, and in that discussion we made use of the derivative notation only. We may now state these laws in terms of differentials.

We shall indicate that an integral is to be taken by placing the symbol $\int$ before the differential involved. Thus, if we have

$$dy = f'(x)\,dx,$$

then we indicate the integral by writing

$$\int dy = \int f'(x)\,dx,$$

which is to be read "the integral of $f'(x)\,dx$." This expression is precisely equivalent to the expression $D_x^{-1} f'(x)$. Thus we may write indifferently

$$D_x^{-1}\,1 \text{ or } \int dx,$$

$$D_x^{-1}\,x^2 \text{ or } \int x^2\,dx,$$

$$D_x^{-1}\sqrt{x^2 - a^2} \text{ or } \int \sqrt{x^2 - a^2}\,dx, \text{ etc.}$$

When written in the differential form, we may speak of the integral also as the *anti-differential.*

From Arts. 39, 40, we may write at once the following important formulas:

1. $\int u^n du = \frac{u^{n+1}}{n+1} + C;\ n \neq -1.$

2. $\int au^n du = a \int u^n du.$

3. $\int (u_1 + u_2 + \ldots + u_n)\, dx = \int u_1 dx + \int u_2 dx + \ldots + \int u_n dx.$

As in Chapter V, these formulas are sufficient to enable us to integrate simple algebraic functions whose integrals are also algebraic. The formulas are, however, general in their application.

EXERCISES.

1. $\int x^3 dx.$
2. $\int (4\,x^3 - 6\,x)\, dx.$
3. $\int (x^2 - 3\,x + 4)\, dx.$
4. $\int \frac{dx}{x^3}.$
5. $\int - 5\,x^{-3}\, dx.$
6. $\int \frac{3\,x - 7}{x^3}\, dx.$
7. $\int \frac{6\,x^3 - 2\,x + 3}{5\,x^4}\, dx.$
8. $\int \left(ax^2 + b + \frac{c}{x^2}\right) dx.$
9. $\int (5\,x^{-3} - 8\,x^{-2} + 3)\, dx.$
10. $\int (a + bt + ct^2)\, dt.$
11. $\int \left(\theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7}\right) d\theta.$

MISCELLANEOUS EXERCISES.

1. Differentiate the following functions, using differentials:

(a) $\frac{x^{2n}}{(1 + x^2)^n}$. (b) $\frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{x}$. (c) $5\,x^{\frac{2}{3}}\,(x^2 - 4)^{\frac{1}{2}}\,(x + 7)^{\frac{1}{3}}$.

2. Find $\frac{dy}{dx}$ for the functions defined by the following equations:

(a) $x^3 - 5\,ax^2y + 7\,y^3 = 0.$ (b) $x^2 - y^2 = \frac{3\,xy}{\sqrt{1 - y^2}}.$

(c) $(ax + by + c)^2 = x^2 + y^2 + 1.$ (d) $x^2y^2 = (a^2 - y^2)\,(b + y)^2.$

3. Perform the following integrations:

(a) $\int x^2(x^3 - 5)\,dx.$ (b) $\int x\sqrt{3x^2 - 7}\,dx.$ (c) $\int \frac{(4x + 10)\,dx}{\sqrt{x^2 + 5x + 3}}.$

(d) $\int (x^{-\frac{5}{2}} - 3x^{-\frac{3}{2}} + 2x^{-\frac{1}{2}} - 5x^{\frac{1}{2}})\,dx.$ x^2+5x+3

4. (a) Classify each of the functions in Ex. 1. (b) Which of these functions have points of discontinuity?

5. Find expressions for the velocity components of a point moving in the parabola $y^2 = 20x$ with a speed of 12 in./sec. Find the values for the point (5, 10).

6. A point moves in the straight line $6x - 8y = 12$ with a velocity of 5 ft./sec. Find the components of the velocity along the X- and Y-axes, respectively.

7. A point moves in the circle $x^2 + y^2 = 36$ with a velocity of 8 ft./sec. Find the X- and Y-components when the point is at $(5, \sqrt{11})$.

8. A point moves along the curve $\rho = \frac{4}{\sqrt{\theta}}$. When $\rho = 3$, the component of the point's velocity along the radius vector is 5 ft./sec. Find (a) the component perpendicular to the radius vector, and (b) the velocity in the curve.

9. A crank pin moves in a circle 2 feet in diameter with a constant speed of 28 ft./sec. When the crank makes an angle of 30° with the horizontal, what is (a) the vertical component of the crank pin's velocity? (b) the horizontal component?

10. Sand or grain, when poured from a height on a level surface, forms a cone with a circular base and a constant angle β at the vertex, dependent on the material. Let a denote the radius of the base of the cone at a given time, and suppose material is being added at the rate of C cu. ft./sec. At what rate is the radius increasing?

11. The velocity of a jet of liquid issuing from an orifice is given by the formula, $v = \sqrt{2gh}$, where h is the height of the liquid surface above the orifice. If $h = 100$ feet and is decreasing at the rate of 0.2 ft./sec., find the rate at which v is decreasing. Take $g = 32.2$.

12. Given $x = 3y^2 + 7y + 1$; find D_xy without first obtaining y as an explicit function of x. $\frac{1}{6y+7}$

13. Plot the curve $y = x^2 \sin \frac{1}{x}$ and discuss its continuity for $x = 0$. How about its derivative for the same value of x?

14. Show that every integral rational algebraic function is continuous and has a derivative.

15. Air expands according to the adiabatic law $pv^{1.4} = C$. When the pressure is 40 lb./sq. in., the volume is 5 cubic feet and is increasing at the rate of 0.2 cu. ft./sec. Find the rate at which the pressure is changing.

16. The part of the perimeter of a cross section of a channel in contact with the water it carries is called the wetted perimeter. With the dimensions given in Fig. 34 the wetted perimeter is 15 feet. Suppose the surface of the water to be falling at the rate of 0.2 in./min. At what rate is the wetted perimeter decreasing?

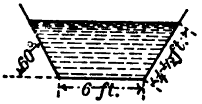

Fig. 34.

17. The efficiency of a Carnot heat engine is given by the formula

$$E = \frac{T - T_0}{T},$$

in which T is the absolute temperature at which the engine receives heat, and T_0 that at which it rejects waste heat. With a fixed value of T_0, find the rate of increase of efficiency compared with the rate of increase of the higher temperature T.

18. The formula for the electrical resistance of a platinum wire is

$$R = R_0 (1 + a\tau + b\tau^2),$$

where R_0, a, and b are constants, and τ denotes the temperature of the wire. Find the rate of increase of resistance at the temperature τ_1, if the temperature is rising at the rate of 0.1° per second.

19. Another formula for the variation of electrical resistance of a metal wire with the temperature is $R = R_0 (1 - c\tau + d\tau^2)^{-1}$. Find the rate of change of the resistance compared with that of the temperature at the temperature τ_1.

20. Given $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$. (a) For what values of x has the curve representing this function a turning point? (b) For what value of x does the tangent make an angle of 45° with the X-axis? (c) Write the equation of the normal at $x = \frac{a}{2}$.

21. Interpret geometrically by means of Fig. 30 the formula for the approximate value of small errors, Art. 33.

CHAPTER VII

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

52. Circular functions. In the present chapter we shall concern ourselves with the development of certain formulas which are of importance in the differentiation and integration of trigonometric, exponential, and logarithmic functions.

At the close of the chapter will be added a list of fundamental formulas for differentiation and integration, and also some applications of the calculus to those problems in physics involving the use of transcendental functions.

The student should recall the following fundamental limits:

$$\underset{\Delta x \doteq 0}{L} \frac{\sin \Delta x}{\Delta x} = 1, \qquad \text{(Art. 11.)}$$

$$\underset{\Delta x \doteq 0}{L} \frac{1 - \cos \Delta x}{\Delta x} = 0, \qquad \text{(Art. 13, Ex. 7.)}$$

which will be of use in deducing the differential formulas for circular functions.

In what follows u is to be regarded as a function of x.

1. *Differentiation of sin u*

Let $\qquad y = \sin u.$

Then $\quad y + \Delta y = \sin (u + \Delta u)$

and $\qquad \Delta y = \sin (u + \Delta u) - \sin u$

$$= \sin u \cos \Delta u + \sin \Delta u \cos u - \sin u.$$

Therefore

$$\frac{\Delta y}{\Delta u} = \left[- \frac{\sin u (1 - \cos \Delta u)}{\Delta u} + \cos u \frac{\sin \Delta u}{\Delta u} \right],$$

and

$$\underset{\Delta u \doteq 0}{L} \frac{\Delta y}{\Delta u} = \left[- \sin u \underset{\Delta u \doteq 0}{L} \frac{1 - \cos \Delta u}{\Delta u} + \cos u \underset{\Delta u \doteq 0}{L} \frac{\sin \Delta u}{\Delta u} \right].$$

But

$$\underset{\Delta u \doteq 0}{L} \frac{1 - \cos \Delta u}{\Delta u} = 0, \text{ and } \underset{\Delta u \doteq 0}{L} \frac{\sin \Delta u}{\Delta u} = 1;$$

hence

$$\underset{\Delta u \doteq 0}{L} \frac{\Delta y}{\Delta u} = D_u y = \cos u.$$

Since u is a function of x, we have by Art. 21

$$D_x y = D_u y \cdot D_x u.$$

Therefore

$$D_x y = \cos u \, D_x u;$$

that is,

$$\boldsymbol{D_x (\sin u) = \cos u \, D_x u.} \tag{1}$$

In the differential notation, we have

$$dy = \cos u \, du,$$

or

$$\boldsymbol{d (\sin u) = \cos u \, du.} \tag{1'}$$

The other circular functions may now be differentiated by applying the general laws of differentiation.

2. Differentiation of cos u.

Let

$$y = \cos u = \sin \left(\frac{\pi}{2} - u\right).$$

Then by (1)

$$D_x y = \cos \left(\frac{\pi}{2} - u\right) D_x \left(\frac{\pi}{2} - u\right)$$

$$= - \sin u \, D_x u;$$

that is,

$$\boldsymbol{D_x (\cos u) = - \sin u \, D_x u,} \tag{2}$$

or, in the notation of differentials,

$$\boldsymbol{d (\cos u) = - \sin u \, du.} \tag{2'}$$

3. Differentiation of tan u.

If

$$y = \tan u = \frac{\sin u}{\cos u},$$

then

$$D_x y = \frac{\cos u \, D_x \sin u - \sin u \, D_x \cos u}{\cos^2 u}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u} D_x u$$

$$= \frac{1}{\cos^2 u} D_x u = \sec^2 u \, D_x u.$$

Hence

$$\boldsymbol{D_x (\tan u) = \sec^2 u \, D_x u,} \tag{3}$$

or

$$\boldsymbol{d (\tan u) = \sec^2 u \, du.} \tag{3'}$$

4. Differentiation of cot u.

Let $$y = \cot u = \frac{\cos u}{\sin u};$$

then $$D_x y = \frac{\sin u \, D_x \cos u - \cos u \, D_x \sin u}{\sin^2 u}$$

$$= -\frac{\sin^2 u + \cos^2 u}{\sin^2 u} D_x u = -\csc^2 u \, D_x u.$$

Hence $$\boldsymbol{D_x (\cot u) = -\csc^2 u \, D_x u,} \tag{4}$$

or $$\boldsymbol{d (\cot u) = -\csc^2 u \, du.} \tag{4'}$$

5. Differentiation of sec u.

If $$y = \sec u = \frac{1}{\cos u},$$

we have $$D_x y = -\frac{1}{\cos^2 u} D_x (\cos u)$$

$$= \sec^2 u \sin u \, D_x u = \sec u \tan u \, D_x u.$$

Therefore $$\boldsymbol{D_x (\sec u) = \sec u \tan u \, D_x u,} \tag{5}$$

and $$\boldsymbol{d (\sec u) = \sec u \tan u \, du.} \tag{5'}$$

6. Differentiation of csc u.

Proceeding as in 5, we find

$$\boldsymbol{D_x (\csc u) = -\csc u \cot u \, D_x u,} \tag{6}$$

or

$$\boldsymbol{d (\csc u) = -\csc u \cot u \, du.} \tag{6'}$$

The details of the proof are left to the student.

7. Differentiation of vers u.

Let $$y = \operatorname{vers} u = 1 - \cos u,$$

then $$D_x y = D_x(1 - \cos u) = \sin u \, D_x u.$$

Hence $$\boldsymbol{D_x (\operatorname{vers} u) = \sin u \, D_x u,} \tag{7}$$

or $$\boldsymbol{d (\operatorname{vers} u) = \sin u \, du.} \tag{7'}$$

Ex. Differentiate $f(x) = \tan^2 \sqrt{a^2 - x^2}$.

We have $$y = \tan^2 \sqrt{a^2 - x^2},$$

whence $$D_x y = 2 \tan \sqrt{a^2 - x^2} \, D_x \tan \sqrt{a^2 - x^2}.$$

From (3), taking $u = \sqrt{a^2 - x^2}$, we obtain

$$D_x \tan\sqrt{a^2 - x^2} = \sec^2\sqrt{a^2 - x^2}\, D_x\sqrt{a^2 - x^2}$$

$$= -\sec^2\sqrt{a^2 - x^2}\,\frac{x}{\sqrt{a^2 - x^2}}\cdot$$

Hence, $$f'(x) = -\frac{2x}{\sqrt{a^2 - x^2}}\tan\sqrt{a^2 - x^2}\sec^2\sqrt{a^2 - x^2}.$$

EXERCISES.

Differentiate the following:

1. $y = \sin ax.$
2. $y = \cos^2 x.$
3. $y = \sin 2x + \cos 2x.$
4. $y = \tan^2 x.$
5. $y = \sin ax \cos bx.$
6. $y = \sin^2 x \tan x.$
7. $y = \sin\frac{x}{2}\cdot$
8. $y = \tan\frac{x}{2}\cdot$
9. $y = x\tan\frac{x}{2}\cdot$
10. $y = \tan\theta - \theta.$
11. $y = \cos 2x.$
12. $y = \frac{1 + \sin x}{1 - \sin x}\cdot$
13. $\rho = a\cos 2\theta.$
14. $\rho = b(1 - \cos\theta).$
15. $\rho = \frac{1}{1 - \cos\theta}\cdot$
16. $\rho = 2a\,\frac{\sin^2\theta}{\cos\theta}\cdot$
17. $\rho = a\sqrt{\cos 2\theta}.$
18. $\rho = a(\sin n\theta)^{\frac{1}{n}}.$
19. $\rho = a\sec^2\frac{\theta}{3}\cdot$
20. $s = a\cos\sqrt{rt}.$
21. $s = A\cos\omega t - B\sin\omega t.$

22. $s = -r\cos\theta + L\left(1 - \sqrt{1 - \frac{r^2}{L^2}\sin^2\theta}\right)\cdot$

23. Find the derivative $D_x y$, when

$$x = a(\phi - \sin\phi),$$
$$y = a(1 - \cos\phi).$$

24. Let
$$x = a\cos\phi + a\phi\sin\phi,$$
$$y = a\sin\phi - a\phi\cos\phi.$$
Find $D_\phi x$, $D_\phi y$, and $D_x y$.

25. Find the polar subtangent and polar subnormal of the curves:

(a) $\rho = \sin\theta.$ (b) $\rho = a(1 - \cos\theta).$ (c) $\rho = a\sec^2\frac{\theta}{2}\cdot$

26. From the equation $\sin 2\theta = 2\sin\theta\cos\theta$, derive by differentiation a formula for $\cos 2\theta$.

27. Find the subtangent and subnormal of the curve $y = a\sin x$.

28. Find the angle at which the curves $y = \cos x$ and $y = \tan x$ intersect.

29. Find the angle which the curves $y = \sin x$ and $y = \cos x$ make with each other at their points of intersections.

30. Find the value of θ for which $\tan\theta$ is increasing twice as fast as θ.

31. When $\theta = 22°$, find approximately the changes in $\sin\theta$ and $\cos\theta$, for a change of 1′ in the angle. (Use radian measure.)

53. Inverse circular functions. The following are the formulas for the differentiation of the inverse circular functions. The proofs for two of these formulas are given; the remaining proofs are left as exercises.

$$\left.\begin{aligned} D_x \operatorname{arc} \sin u &= \frac{1}{\sqrt{1-u^2}} D_x u; \\ d \operatorname{arc} \sin u &= \frac{du}{\sqrt{1-u^2}}. \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} D_x \operatorname{arc} \cos u &= -\frac{1}{\sqrt{1-u^2}} D_x u; \\ d \operatorname{arc} \cos u &= -\frac{du}{\sqrt{1-u^2}}. \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} D_x \operatorname{arc} \tan u &= \frac{1}{1+u^2} D_x u; \\ d \operatorname{arc} \tan u &= \frac{du}{1+u^2}. \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} D_x \operatorname{arc} \cot u &= -\frac{1}{1+u^2} D_x u; \\ d \operatorname{arc} \cot u &= -\frac{du}{1+u^2}. \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} D_x \operatorname{arc} \sec u &= \frac{1}{u\sqrt{u^2-1}} D_x u; \\ d \operatorname{arc} \sec u &= \frac{du}{u\sqrt{u^2-1}}. \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} D_x \operatorname{arc} \csc u &= -\frac{1}{u\sqrt{u^2-1}} D_x u; \\ d \operatorname{arc} \csc u &= -\frac{du}{u\sqrt{u^2-1}}. \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} D_x \operatorname{arc} \operatorname{vers} u &= \frac{1}{\sqrt{2u-u^2}} D_x u; \\ d \operatorname{arc} \operatorname{vers} u &= \frac{du}{\sqrt{2u-u^2}}. \end{aligned}\right\} \tag{7}$$

Proof of (1).

Let $y = \text{arc sin } u,$

then $u = \sin y.$

Differentiating, we obtain

$$D_y u = \cos y.$$

Therefore $D_u y = \dfrac{1}{\cos y},$

and $D_x y = D_u y \cdot D_x u$ (Art. 21)

$$= \frac{1}{\cos y} D_x u.$$

Since $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - u^2},$

we have $D_x y = \dfrac{1}{\sqrt{1 - u^2}} D_x u.$

Proof of (5).

Let $y = \text{arc sec } u,$

then $u = \sec y,$

and $D_y u = \sec y \tan y.$

Hence $D_u y = \dfrac{1}{\sec y \tan y},$

and $D_x y = D_u y \, D_x u$ (Art. 21)

$$= \frac{1}{\sec y \tan y} D_x u.$$

Since $\sec y = u,$ and $\tan y = \sqrt{\sec^2 y - 1} = \sqrt{u^2 - 1},$

we may write $D_x y = \dfrac{1}{u\sqrt{u^2 - 1}} D_x u.$

It will be seen that the derivatives of all the inverse circular functions except arc tan u and arc cot u contain radicals. Since the sign of the radical may be taken either positive or negative, the sign of the derivative containing it is ambiguous. This ambiguity may be explained as follows: If $y = \text{arc sin } x$, then y is a multiple-valued function of x. For some of these values the derivative is positive, while for others it is negative. Thus in the graph of $y =$ arc sin x shown in Fig. 35, we have different values of y as LM_1, LM_2, LM_3, etc., for $x = OL$. At M_1, M_3, as indicated by the slope of the curve, the derivative is positive; at M_2 it is negative. To make the inverse functions single-valued, we restrict the angle y to values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for the inverse sine, cosecant, and tangent, and between 0 and π for the inverse cosine, cotangent, and secant. With these restrictions the radicals in the preceding formulas are positive except in (5) and (6); and they are positive in (5) and (6) if $0 < y < \frac{1}{2}\pi$.

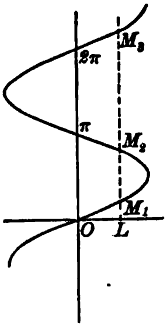

Fig. 35.

It will be observed that the derivatives of the inverse circular functions are algebraic, not transcendental.

Ex. 1.

$$y = \text{arc tan}\,\frac{x}{\sqrt{1 - 2x^2}}\cdot$$

Let $$\frac{x}{\sqrt{1-2x^2}} = u;\ \text{then } y = \text{arc tan}\,u,\quad dy = \frac{du}{1+u^2},$$

and $$du = d\,\frac{x}{\sqrt{1-2x^2}} = \frac{\sqrt{1-2x^2} + \dfrac{2x^2}{\sqrt{1-2x^2}}}{1-2x^2}\,dx = \frac{1}{(1-2x^2)^{\frac{3}{2}}}\,dx.$$

Substituting for u and du, we get

$$dy = \frac{1}{1 + \dfrac{x^2}{1-2x^2}}\cdot\frac{1}{(1-2x^2)^{\frac{3}{2}}}\,dx = \frac{1}{(1-x^2)\sqrt{1-2x^2}}\,dx.$$

Ex. 2. $$y = x\,\text{arc cos}\,\frac{x}{a}\cdot$$

$$dy = \text{arc cos}\,\frac{x}{a}\,dx + x\,d\,\text{arc cos}\,\frac{x}{a} = \text{arc cos}\,\frac{x}{a}\,dx - \frac{x}{\sqrt{a^2-x^2}}\,dx,$$

or $$\frac{dy}{dx} = \text{arc cos}\,\frac{x}{a} - \frac{x}{\sqrt{a^2-x^2}}\cdot$$

EXERCISES.

Differentiate the following:

1. $y = \text{arc sin } \frac{x}{a}.$
2. $y = \text{arc tan } \frac{x}{\sqrt{k}}.$
3. $y = \text{arc csc } \frac{x^2}{a^2}.$
4. $y = x \text{ arc tan } x.$
5. $y = \text{arc cos } \frac{3x - 2}{5}.$
6. $y = \text{arc sin } (2x\sqrt{1 - x^2}).$
7. $y = \text{arc tan } \frac{x}{a} + \text{arc tan } \frac{a}{x}.$
8. $y = \text{arc cos } (\cos x).$
9. $s = \text{arc sec } \frac{1}{\sqrt{1 - t^2}}.$
10. $s = \text{arc cos } \sqrt{\frac{1 - t^2}{1 + t^2}}.$
11. $y = \text{arc tan } \sqrt{1 - k^2 \tan x}.$
12. $y = \text{arc vers } \frac{3x^2}{a^2}.$
13. $y = \frac{1}{2a(x - a)} + \frac{1}{2a^2} \text{ arc tan } \frac{x}{a}.$
14. $y = \frac{1}{mn} \text{ arc tan } \frac{x}{m}.$

54. Exponential and logarithmic functions. The differentiation of the exponential functions a^u and e^u depends upon the evaluation of the limit

$$\underset{x \doteq 0}{L} \frac{a^x - 1}{x},$$

which was discussed in Chapter II, Art. 14.

1. Differentiation of a^u.

Let $$y = a^u;$$

then $$y + \Delta y = a^{u+\Delta u},$$

and $$\Delta y = a^{u+\Delta u} - a^u = a^u (a^{\Delta u} - 1).$$

Therefore $$\frac{\Delta y}{\Delta u} = a^u \frac{a^{\Delta u} - 1}{\Delta u}.$$

Taking limits, we have

$$\underset{\Delta u \doteq 0}{L} \frac{\Delta y}{\Delta u} = a^u \underset{\Delta u \doteq 0}{L} \frac{a^{\Delta u} - 1}{\Delta u}.$$

But $$\underset{\Delta u \doteq 0}{L} \frac{a^{\Delta u} - 1}{\Delta u} = \log a; \qquad \text{[Art. 14 (e)]}$$

hence $$D_u y = \underset{\Delta u \doteq 0}{L} \frac{\Delta y}{\Delta u} = a^u \log a.$$

Using the theorem of Art. 21, we have

$$D_x y = D_u y \cdot D_x u.$$

Therefore $$D_x y = a^u \log a \, D_x u;$$

that is,

or $$\left.\begin{aligned} D_x a^u &= a^u \log a \; D_x u \\ da^u &= a^u \log a \, du. \end{aligned}\right\} \qquad (1)$$

If $u = x$, we have

$$D_x a^x = a^x \log a, \quad \text{or} \quad da^x = a^x \log a \, dx. \qquad (1')$$

2. Differentiation of e^u.

If in (1) we substitute e for a, we obtain (since $\log e = 1$)

$$D_x e^u = e^u D_x u; \quad de^u = e^u du. \qquad (2)$$

For $u = x$, we have

$$D_x e^x = e^x; \qquad de^x = e^x dx. \qquad (2')$$

It may be remarked that e^x is a function whose derivative is equal to the function itself.

3. Differentiation of $\log_a u$.

Let $$y = \log_a u,$$

whence $$u = a^y.$$

From (1) we obtain

$$D_y u = a^y \log a,$$

whence $$D_u y = \frac{1}{a^y \log a}.$$

By the theorem of Art. 21,

$$D_x y = D_u y \cdot D_x u$$

$$= \frac{1}{a^y \log a} D_x u = \frac{1}{u \log a} D_x u.$$

We have therefore

$$D_x \log_a u = \frac{1}{\log a} \frac{D_x u}{u}, \text{ or } d \log_a u = \frac{1}{\log a} \frac{du}{u}. \qquad (3)$$

If $u = x$, we have

$$D_x \log_a x = \frac{1}{\log a} \frac{1}{x}; \qquad d \log_a x = \frac{1}{\log a} \frac{dx}{x}. \qquad (3')$$

The fraction $\frac{1}{\log a}$ is called the *modulus* of the system of logarithms whose base is a, and is denoted by m. For the Briggs' or common system, in which $a = 10$, $m = .434294.....$

We have, therefore,

$$D_x \log_a x = \frac{m}{x}.$$

4. Differentiation of log u.

By the substitution of e for a in (3) and (3′), the following formulas are obtained (since $\log e = 1$):

$$\boldsymbol{D_x \log u = \frac{1}{u} D_x u; \qquad d \log u = \frac{du}{u}.} \tag{4}$$

$$\boldsymbol{D_x \log x = \frac{1}{x}; \qquad d \log x = \frac{dx}{x}.} \tag{4′}$$

Ex. 1.
$$y = \log \sin x.$$
$$dy = \frac{1}{\sin x}\, d \sin x = \frac{1}{\sin x} \cos x\, dx = \cot x\, dx.$$

Ex. 2.
$$y = e^{\sqrt{1+x^2}}.$$
$$dy = e^{\sqrt{1+x^2}}\, d(\sqrt{1+x^2}) = e^{\sqrt{1+x^2}} \frac{x}{\sqrt{1+x^2}} dx.$$

55. Logarithmic differentiation. If a function consists of a number of factors, it may be conveniently differentiated by taking logarithms. Thus, if

$$y = uvw,$$
$$\log y = \log u + \log v + \log w.$$

Differentiating, we have

$$\frac{dy}{y} = \frac{du}{u} + \frac{dv}{v} + \frac{dw}{w},$$

or

$$dy = \frac{y}{u} du + \frac{y}{v} dv + \frac{y}{w} dw.$$

Ex. 1.
$$y = \sqrt{\frac{(x-a)(x-b)}{x-c}}.$$

Taking logarithms, we have

$$\log y = \tfrac{1}{2} [\log (x-a) + \log (x-b) - \log (x-c)].$$

Differentiating, we get

$$\frac{dy}{y} = \frac{1}{2} \left[\frac{dx}{x-a} + \frac{dx}{x-b} - \frac{dx}{x-c}\right],$$

whence

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-a} + \frac{1}{x-b} - \frac{1}{x-c}\right].$$

Substituting for y from the original equation, the final result is

$$\frac{dy}{dx} = \frac{1}{2}\left[\sqrt{\frac{x-b}{(x-a)(x-c)}} + \sqrt{\frac{x-a}{(x-b)(x-c)}} - \sqrt{\frac{(x-a)(x-b)}{(x-c)^3}}\right].$$

Logarithmic differentiation is especially useful in differentiating an exponential function having a variable base.

Ex. 2.
$$y = e^x x^{\frac{1}{x}}.$$

$$\log y = \log e^x + \frac{1}{x}\log x$$

$$= x + \frac{1}{x}\log x,$$

$$\frac{dy}{y} = dx - \frac{1}{x^2}\log x\,dx + \frac{1}{x^2}\,dx,$$

$$\frac{dy}{dx} = y\left[1 + \frac{1}{x^2}(1 - \log x)\right]$$

$$= e^x x^{\frac{1}{x}} + e^x x^{\frac{1}{x}-2}(1 - \log x).$$

Logarithmic differentiation also enables us to differentiate u^n, where n may have any constant value. We have

$$y = u^n,$$

$$\log y = n \log u.$$

Differentiating, we obtain

$$\frac{1}{y}D_x y = \frac{n}{u}D_x u,$$

whence
$$D_x y = n\frac{y}{u}D_x u,$$

or
$$D_x y = nu^{n-1}D_x u.$$

This completes the demonstration begun in Art. 20, and shows that the formula given there for the derivative of u^n holds for all *real* values of n.

EXERCISES.

1. $y = \log(x^2 - 3x + 5)$.
2. $y = \log_a\left(\dfrac{x+m}{x-m}\right)$.
3. $y = a^{\sqrt{x^2 - a^2}}$.
4. $y = e^{x^2} + e^x$.
5. $y = x \log x$.
6. $y = \log(\log x)$.
7. $y = e^x + e^{-x}$.
8. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.
9. $y = e^{\sin x}$.
10. $y = \log \tan x$.
11. $y = \log \dfrac{\sqrt{x^2+1} - x}{\sqrt{x^2+1} + x}$.

In Ex. 12 to 15, differentiate by taking logarithms:

12. $y = x(1 - x)\sqrt{1 + x^2}$.
13. $y = \dfrac{(x-5)^{\frac{2}{3}}(3x-4)^{\frac{1}{4}}}{(2x+5)^{\frac{3}{2}}}$.
14. $y = \dfrac{3x^2 - 1}{x\sqrt{1 - x^2}}$.
15. $y = \sqrt{\dfrac{1-x}{1+x}}$.

16. If $z = Ae^{\theta\sqrt{k}} + Be^{-\theta\sqrt{k}}$, find $\dfrac{dz}{d\theta}$.

17. If $s = (A + Bt)e^{-kt}$, find $\dfrac{ds}{dt}$.

18. If $s = e^{-kt}[a \sin mt + b \cos mt]$, find $\dfrac{ds}{dt}$.

19. Find the polar subtangent, polar subnormal, lengths of polar tangent and polar normal of the curve $\rho = e^{a\theta}$.

20. Find the angle between the curve $y = \log x$ and the axis of x; between the same curve and the line $y = 2$.

21. Find the curve whose subtangent (rectangular coördinates) is constant.

22. Find the subtangent, subnormal, and normal of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

23. Given $\log 4.32 = 1.4633$, find approximately the value of $\log 4.33$ by means of the theorem $\Delta y = f'(x)\,\Delta x$.

24. By logarithmic differentiation prove formula (1), Art. 54.

56. The hyperbolic and inverse hyperbolic functions. In many of the applications of calculus, it is convenient to use certain functions which have somewhat the same relation to the equilateral hyperbola that the trigonometrical functions have to the circle. They are called the *hyperbolic functions*, and may be defined in terms of the exponential functions as follows:

Name.	Symbol.	Definition.
Hyperbolic sine,	$\sinh x$	$= \frac{1}{2}(e^x - e^{-x})$.
Hyperbolic cosine,	$\cosh x$	$= \frac{1}{2}(e^x + e^{-x})$.
Hyperbolic tangent,	$\tanh x$	$= \dfrac{\sinh x}{\cosh x}$.
Hyperbolic cotangent,	$\coth x$	$= \dfrac{\cosh x}{\sinh x}$.
Hyperbolic secant,	$\operatorname{sech} x$	$= \dfrac{1}{\cosh x}$.
Hyperbolic cosecant,	$\operatorname{csch} x$	$= \dfrac{1}{\sinh x}$.

The student may derive the following relations between the hyperbolic functions:

$$\cosh^2 x - \sinh^2 x = 1. \tag{1}$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x. \tag{2}$$

$$\operatorname{csch}^2 x = \coth^2 x - 1. \tag{3}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y. \tag{4}$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y. \tag{5}$$

$$\sinh 2x = 2 \sinh x \cosh x. \tag{6}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x. \tag{7}$$

The following formulas give the derivatives of the hyperbolic functions. The first two are derived by differentiating the defining exponential equations; the others by applying the general theorems of differentiation.

$$D_x \sinh u = \cosh u \, D_x u. \tag{8}$$

$$D_x \cosh u = \sinh u \, D_x u. \tag{9}$$

$$D_x \tanh u = \operatorname{sech}^2 u \, D_x u. \tag{10}$$

$$D_x \coth u = -\operatorname{csch}^2 u \, D_x u. \tag{11}$$

$$D_x \operatorname{sech} u = -\tanh u \operatorname{sech} u \, D_x u. \tag{12}$$

$$D_x \operatorname{csch} u = -\coth u \operatorname{csch} u \, D_x u. \tag{13}$$

As we have inverse circular functions, so we have inverse hyperbolic functions; thus, if $y = \cosh x$, the inverse function is given by $x = \text{inv.} \cosh y$, just as $x = \text{arc} \cos y$ when $y = \cos x$.

Each of these functions can be expressed in terms of a logarithmic function.

Thus, if

$$y = \text{inv. } \sinh x,$$

$$x = \sinh y = \tfrac{1}{2}(e^y - e^{-y}),$$

whence

$$e^{2y} - 2\,x\,e^y - 1 = 0.$$

Solving the quadratic in e^y, we get

$$e^y = x \pm \sqrt{x^2 + 1},$$

or

$$y = \log\,(x \pm \sqrt{x^2 + 1}).$$

In order to make the quantity within the parenthesis positive, the upper sign of the radical must be taken; hence

$$y = \text{inv. } \sinh x = \log\,(x + \sqrt{x^2 + 1}). \qquad (14)$$

The student may in a similar manner derive the following results:

$$\text{inv. } \cosh x = \log\,(x \pm \sqrt{x^2 - 1}),\ (x^2 > 1). \qquad (15)$$

$$\text{inv. } \tanh x = \log \sqrt{\frac{1 + x}{1 - x}},\ (x^2 < 1). \qquad (16)$$

$$\text{inv. } \coth x = \log \sqrt{\frac{x + 1}{x - 1}},\ (x^2 > 1). \qquad (17)$$

By using the method of Art. 53, the derivatives of the inverse hyperbolic functions may be found. Thus

$$D_x \text{ inv. } \sinh u = \frac{1}{\sqrt{u^2 + 1}}\,D_x u. \qquad (18)$$

$$D_x \text{ inv. } \cosh u = \pm\frac{1}{\sqrt{u^2 - 1}}\,D_x u. \qquad (19)$$

$$D_x \text{ inv. } \tanh u = \frac{1}{1 - u^2}\,D_x u. \qquad (20)$$

EXERCISES.

1. Draw the graphs of e^x and e^{-x}, and from these by addition and subtraction of ordinates draw the graphs of cosh x and sinh x.

2. Show that tanh x lies between -1 and $+1$, and draw its graph.

3. Verify the following differentiations:

$$y = \sinh^2 x, \qquad D_x y = \sinh 2x.$$

$$y = \tanh^2 x, \qquad \frac{dy}{dx} = \frac{2 \sinh x}{\cosh^3 x}.$$

$$y = \log \cosh x, \qquad dy = \tanh x\,dx.$$

$$y = x - \tanh x, \qquad dy = \tanh^2 x\,dx.$$

4. Differentiate $y = \frac{1}{3}\cosh^3 x - \cosh x$.

5. Differentiate $y = \log\cosh x - \frac{1}{2}\tanh^2 x$.

6. The equation of the catenary is $y = a\cosh\frac{x}{a}$. Find (a) the equation of the tangent at the point (x_1, y_1); (b) the subtangent.

57. Fundamental differentials and integrals. From the results obtained in the preceding sections of this chapter, we have the following list of fundamental differentials and integrals. For the sake of completeness, the algebraic integral is included. The student should make himself thoroughly familiar with these formulas.

1. $d(u^n) = nu^{n-1}\,du,$ $\qquad \int u^n du = \frac{u^{n+1}}{n+1} + C. \quad (n \neq -1)$

2. $d(\sin u) = \cos u\,du,$ $\qquad \int \cos u\,du = \sin u + C.$

3. $d(\cos u) = -\sin u\,du,$ $\qquad \int \sin u\,du = -\cos u + C.$

4. $d(\tan u) = \sec^2 u\,du,$ $\qquad \int \sec^2 u\,du = \tan u + C.$

5. $d(\cot u) = -\csc^2 u\,du,$ $\qquad \int \csc^2 u\,du = -\cot u + C.$

6. $d(\sec u) = \sec u\tan u\,du,$ $\qquad \int \sec u\tan u\,du = \sec u + C.$

7. $d(\csc u) = -\csc u\cot u\,du,$ $\qquad \int \csc u\cot u\,du = -\csc u + C.$

8. $d(\text{arc}\sin u) = -d(\text{arc}\cos u) = \frac{du}{\sqrt{1-u^2}},$ $\qquad \int \frac{du}{\sqrt{1-u^2}} = \text{arc}\sin u + C = -\text{arc}\cos u + C.$

9. $d(\text{arc}\tan u) = -d(\text{arc}\cot u) = \frac{du}{1+u^2},$ $\qquad \int \frac{du}{1+u^2} = \text{arc}\tan u + C = -\text{arc}\cot u + C.$

10. $d(\text{arc}\sec u) = -d(\text{arc}\csc u) = \frac{du}{u\sqrt{u^2-1}},$ $\qquad \int \frac{du}{u\sqrt{u^2-1}} = \text{arc}\sec u + C = -\text{arc}\csc u + C.$

11. $d(\text{arc vers}\,u) = \frac{du}{\sqrt{2u-u^2}},$ $\qquad \int \frac{du}{\sqrt{2u-u^2}} = \text{arc vers}\,u + C.$

12. $d(a^u) = a^u \log a \, du,$ $\qquad \int a^u du = \frac{a^u}{\log a} + C.$

13. $d(e^u) = e^u du,$ $\qquad \int e^u du = e^u + C.$

14. $d(\log u) = \frac{du}{u},$ $\qquad \int \frac{du}{u} = \log u + C.$

The following additional integrals are of great importance and should also be committed to memory.

15. $$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{u-a}{u+a} + C. \quad (u^2 > a^2)$$
$$= \frac{1}{2a} \log \frac{a-u}{a+u} + C. \quad (u^2 < a^2).$$

16. $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log (u + \sqrt{u^2 \pm a^2}) + C.$$

17. $$\int \tan u \, du = \log \sec u + C.$$

18. $$\int \cot u \, du = \log \sin u + C.$$

19. $$\int \csc u \, du = \log \tan \frac{u}{2} + C.$$

20. $$\int \sec u \, du = \log \tan \left(\frac{u}{2} + \frac{\pi}{4}\right) + C.$$

The derivation of these last forms will be given later. The student may here verify them by differentiation.

58. Integration by inspection. The integrals tabulated in the preceding article are called *fundamental* or *standard integrals.* In these, the integrands are recognized as the results of previous differentiations; hence, if a given integrand has one of these forms, the integral is known at once as the function previously differentiated. When the integrand has not one of these standard forms, it must be prepared for integration; that is, by suitable transformations, it must be reduced if possible to an integrand of standard form or to a sum of such integrands. The general method used for such reductions forms the subject matter of Chapter XI. In the present chapter we shall consider only such reductions as are quite obvious.

It should be noted that the variable u in the standard forms may be any function, $f(x)$. For example, take the integral

$$\int e^{\sin x}\cos x\, dx.$$

Since $\cos x\, dx = d\,(\sin x)$, this may be written

$$\int e^{\sin x} d\,(\sin x),$$

which will be recognized as form 13, with u replaced by $\sin x$.

Frequently the integrand may be made to assume a standard form by the introduction of a constant factor. The following examples are illustrative:

Ex. **1.** $\int (3\,x - 2)^2\, dx.$

Here the standard form $\int u^2 du$, where $u = 3\,x - 2$, suggests itself. Since $du = d\,(3\,x - 2) = 3\,dx$, the reduction is effected by introducing the factor 3 in the integrand and the neutralizing factor $\frac{1}{3}$ outside of the integral sign. Thus,

$$\int (3\,x - 2)^2\, dx = \tfrac{1}{3}\int (3\,x - 2)^2\, 3\, dx = \tfrac{1}{3}\int (3\,x - 2)^2\, d\,(3\,x - 2) = \tfrac{1}{9}\,(3\,x - 2)^3 + C.$$

Ex. **2.** $\displaystyle\int e^{ax+b} dx = \frac{1}{a}\int e^{ax+b} a\, dx = \frac{1}{a}\int e^{ax+b} d\,(ax + b) = \frac{e^{ax+b}}{a} + C.$

Here the introduction of the factor a reduces the integrand to form 13, u being replaced by $ax + b$.

Ex. **3.** $$\int \frac{x\, dx}{\sqrt{a - bx^2}} = \int (a - bx^2)^{-\frac{1}{2}} x\, dx.$$

Introducing the factor $-2\,b$, we get

$$-\frac{1}{2\,b}\int (a - bx^2)^{-\frac{1}{2}}\,(-2\,bx\, dx) = -\frac{1}{2\,b}\int (a - bx^2)^{-\frac{1}{2}}\, d\,(a - bx^2).$$

This integral will be recognized as form 1 with $u = a - bx^2$, and $n = -\frac{1}{2}$ The solution is therefore

$$-\frac{1}{2\,b}\cdot\frac{(a - bx^2)^{-\frac{1}{2}+1}}{-\frac{1}{2} + 1} + C = -\frac{1}{b}\,(a - bx^2)^{\frac{1}{2}} + C.$$

Ex. **4.** $$\int \frac{dx}{a^2 + b^2x^2}.$$

Looking among the standard forms, it appears that form 9 is the one to which the integrand may be reduced. To get 1 as the first term of the denominator, we divide the denominator by a^2, and thus obtain

$$\frac{1}{a^2}\int \frac{dx}{1 + \frac{b^2}{a^2}\,x^2} = \frac{1}{a^2}\int \frac{dx}{1 + \left(\frac{b}{a}\,x\right)^2}.$$

We observe now that $\frac{b}{a}x$ is the variable, whence the numerator must take the form $d\left(\frac{b}{a}x\right) = \frac{b}{a}\,dx$. Therefore introducing the factor $\frac{b}{a}$, we have

$$\frac{a}{b}\cdot\frac{1}{a^2}\int\frac{\frac{b}{a}\,dx}{1+\left(\frac{b}{a}x\right)^2} = \frac{1}{ab}\,\text{arc tan}\,\frac{bx}{a} + C.$$

Ex. 5. $$\int\frac{(x-1)\,dx}{3x^2-6x+5}.$$

It will be observed that $d(3x^2-6x+5) = 6(x-1)\,dx$; hence except for the factor 6, the numerator of the integrand is the differential of the denominator. By the introduction of this factor the integral reduces to form 14, namely, $\int\frac{du}{u}$. Thus

$$\int\frac{(x-1)\,dx}{3x^2-6x+5} = \frac{1}{6}\int\frac{6(x-1)\,dx}{3x^2-6x+5} = \frac{1}{6}\int\frac{d(3x^2-6x+5)}{3x^2-6x+5}$$
$$= \frac{1}{6}\log(3x^2-6x+5).$$

In some cases the integral reduces to the algebraic sum of several standard integrals.

Ex. 6. $$\int\frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx = \int\frac{1+x}{\sqrt{1-x^2}}\,dx = \int\frac{dx}{\sqrt{1-x^2}} + \int\frac{x\,dx}{\sqrt{1-x^2}}.$$

Now $$\int\frac{dx}{\sqrt{1-x^2}} = \text{arc sin}\,x, \quad \text{(Form 8.)}$$

and $$\int\frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}(-2x\,dx) = -\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}\,d(1-x^2)$$
$$= -(1-x^2)^{\frac{1}{2}}. \quad \text{(Form 1.)}$$

Hence $$\int\frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx = \text{arc sin}\,x - \sqrt{1-x^2} + C.$$

If the denominator of the integrand is of the same degree as the numerator, or of lower degree, divide before integrating.

Ex. 7. $$\int\frac{x^2-7}{x-3}\,dx.$$

We have $$\frac{x^2-7}{x-3} = x + 3 + \frac{2}{x-3}.$$

Hence $$\int\frac{x^2-7}{x-3}\,dx = \int x\,dx + 3\int dx + 2\int\frac{dx}{x-3}$$
$$= \tfrac{1}{2}x^2 + 3x + 2\log(x-3) + C.$$

EXERCISES.

In the following examples determine by inspection the proper standard form, find the function that replaces u in that form, and perform the integration.

1. $\int (x + a)^4\, dx.$
4. $\int 2\,x\, e^{x^2}\, dx.$
7. $\int \frac{\arctan x\, dx}{1 + x^2}.$
2. $\int 2\,x\,(x^2 - a^2)^{\frac{1}{2}}\, dx.$
5. $\int \frac{\cos\theta\, d\theta}{\sin^2\theta}.$
8. $\int \frac{m\, dx}{\sqrt{1 - m^2x^2}}.$
3. $\int \frac{6\,x\, dx}{3\,x^2 + a^2}.$
6. $\int \cos^2\theta \sin\theta\, d\theta.$
9. $\int \sec^2\theta \tan\theta\, d\theta.$

10. Explain why the integration in Ex. 9 may lead to either $\frac{1}{2}\sec^2\theta$ or $\frac{1}{2}\tan^2\theta.$ $\sec^2\vartheta - 1$

In the following, reduce the integrands to standard forms by the introduction of proper factors, and integrate.

11. $\int (ax + b^2)^{\frac{5}{2}}\, dx.$
13. $\int \frac{x\, dx}{\sqrt{a^2 + x^2}}.$
15. $\int \frac{dx}{x\sqrt{a^2x^2 - 1}}.$
12. $\int \frac{x^2\, dx}{2\,x^3 - 5}.$
14. $\int \frac{dx}{\sqrt{1 - c^2x^2}}.$
16. $\int a^{m^2x^2} x\, dx.$ $+ C$

Verify the following important integrations.

17. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C = -\arccos\frac{x}{a} + C'.$

18. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arc\,sec} \frac{x}{a} + C = -\frac{1}{a} \operatorname{arc\,csc} \frac{x}{a} + C'.$

19. $\int \frac{dx}{\sqrt{2ax - x^2}} = \operatorname{arc\,vers}\frac{x}{a} + C.$

59. Integration by substitution. The reduction of a given integrand to a standard form is sometimes most easily effected by the substitution of a new variable. The following examples illustrate this method.

Ex. 1. $$\int \frac{dx}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}.$$

Let $e^{\frac{x}{2}} = z$; then $dx = 2\frac{dz}{z}$, and the integral takes the form

$$\int \frac{2\,dz}{z(z + z^{-1})} = 2\int \frac{dz}{z^2 + 1} = 2 \arctan z + C = 2 \arctan e^{\frac{x}{2}} + C.$$

Ex. 2. $$\int \frac{dx}{x^2\sqrt{1 - x^2}}.$$

Let $x = \cos\theta$; then $dx = -\sin\theta\, d\theta$, $\sqrt{1 - x^2} = \sin\theta$, and the integral takes the form

$$-\int \frac{\sin\theta\, d\theta}{\cos^2\theta \sin\theta} = -\int \sec^2\theta\, d\theta = -\tan\theta + C = -\frac{\sqrt{1 - x^2}}{x} + C.$$

The substitution $x = \frac{1}{z}$ may also be used. The student may work out the details.

As additional exercises, the derivations of forms 15 to 20 are given.

Ex. 3. Since $\frac{1}{u^2 - a^2} = \frac{1}{2a}\left(\frac{1}{u-a} - \frac{1}{u+a}\right)$, we have, if $u^2 > a^2$,

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\int \frac{du}{u-a} - \frac{1}{2a}\int \frac{du}{u+a}$$

$$= \frac{1}{2a}[\log(u-a) - \log(u+a)] = \frac{1}{2a}\log\frac{u-a}{u+a}.$$

The student may derive the second integral of form 15 and show that it applies when $u^2 < a^2$.

Ex. 4. To integrate $\int \frac{du}{\sqrt{u^2 \pm a^2}}$, put $u^2 \pm a^2 = z^2$.

Then $2u\,du = 2z\,dz$, whence $\frac{du}{z} = \frac{dz}{u} = \frac{du + dz}{u + z}$.

Therefore, $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \int \frac{du}{z} = \int \frac{du + dz}{u + z} = \log(u + z)$

$$= \log\left(u + \sqrt{u^2 \pm a^2}\right).$$

Ex. 5. $\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = -\int \frac{d(\cos x)}{\cos x}$

$$= -\log\cos x + C = \log\sec x + C.$$

Ex. 6. $\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{d(\sin x)}{\sin x} = \log\sin x + C.$

Ex. 7. $\int \csc x\,dx.$

Let $z = \tan\frac{x}{2}$; then $\csc x = \frac{1 + z^2}{2z}$, and $dx = \frac{2\,dz}{1 + z^2}$.

Substituting these values, we get

$$\int \csc x\,dx = \int \frac{dz}{z} = \log z + C = \log\tan\frac{x}{2} + C.$$

Ex. 8. $\int \sec x\,dx.$

Making the same substitution as in Ex. 7, the integral reduces to the form

$$\int \frac{2\,dz}{1 - z^2}.$$

Hence

$$\int \sec x\,dx = 2\int \frac{dz}{1-z^2} = \log\frac{1+z}{1-z} + C = \log\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} + C$$

$$= \log\tan\left[\frac{x}{2} + \frac{\pi}{4}\right] + C.$$

From the following examples, the student will notice that it is often possible to reduce the given integrand to a standard form by writing the denominator as the sum or difference of two squares.

Ex. 9. $$\int \frac{dx}{\sqrt{7+6x-x^2}} = \int \frac{d\,(x-3)}{\sqrt{4^2-(x-3)^2}} = \text{arc}\sin\frac{x-3}{4} + C.$$

Ex. 10. $$\int \frac{dx}{5-4x-x^2} = \int \frac{d\,(x+2)}{3^2-(x+2)^2}$$

$$= \frac{1}{6}\log\frac{5+x}{1-x} + C, \quad \text{if } (x+2)^2 < 3^2.$$

$$= \frac{1}{6}\log\frac{x+5}{x-1} + C, \quad \text{if } (x+2)^2 > 3^2.$$

Ex. 11. $$\int \frac{dx}{\sqrt{x^2+4x+8}} = \int \frac{d\,(x+2)}{\sqrt{(x+2)^2+4}}$$

$$= \log\left(x+2+\sqrt{x^2+4x+8}\right) + C.$$

There is a fundamental difference between differentiation and integration. The former is a direct, the latter an inverse process. As we have seen, when once the derivatives of the elementary functions have been found, the derivatives of any functions expressed in terms of these functions can usually be obtained by one or more direct operations. On the contrary, it is not, in general, possible to determine the integral of a function from the known integrals of the elementary functions in terms of which the given function is expressed. There are no general methods for expressing the integral of a product or a quotient of functions, or of a function of a function in terms of the integrals of the component functions. Hence the process of integration consists, not in a series of direct operations carried out in accordance with a general method, but rather in attempts to reduce the given function to a form, the integral of which is known.

Integration, then, is largely a question of judgment in attacking the problem. Specific rules cannot be given, but the following general directions will be found useful. First inspect the integrand carefully and determine whether it is a standard form. Frequently the integrand is merely a disguised standard form and needs only a rearrangement of its terms to be recognized as such. If the integrand is not in a standard form, see if it cannot be made to assume such a form by the introduction of a constant factor. Do not neglect the neutralizing factor outside the integral sign. If this device does not appear to be effective, try the substitution of a new variable.

In the following exercises the student should in each example state the standard form or forms finally used.

EXERCISES.

1. $\int 3\,mx^{c-2}\,dx.$

2. $\int \frac{x^2\,dx}{(m^3+x^3)^{\frac{1}{2}}}.$

3. $\int (5\,x-x^2)^{-\frac{1}{2}}(5-2\,x)\,dx.$

4. $\int \frac{(x-2)\,dx}{\sqrt{x^2-4x+5}}.$

5. $\int e^{x^2}\,x\,dx.$

6. $\int \sin^3\theta\cos\theta\,d\theta$

7. $\int \frac{\cos\theta\,d\theta}{\sin^4\theta}.$

8. $\int \cos(x+a)\,dx.$

9. $\int \frac{(3+5\,x)\,dx}{\sqrt{9-x^2}}.$

10. $\int (ax+b)^{\frac{1}{2}}\,dx.$

11. $\int \frac{x^2\,dx}{4\,x^3+7}.$

12. $\int \frac{(2\,x^3+3\,x^2-4\,x-3)\,dx}{x^4+2\,x^3-4\,x^2-6\,x-10}.$

13. $\int \frac{dx}{\sqrt{x^2+4\,x+13}}.$

14. $\int \frac{x^2\,dx}{x+1}.$

15. $\int \frac{2\,x^2+1}{x^3+x}\,dx.$

16. $\int \frac{\sin x\,dx}{a+b\cos x}.$

17. $\int e^{\cos x}\sin x\,dx.$

18. $\int \frac{1-x}{1+x^2}\,dx.$

19. $\int \frac{dx}{1+3\,x+2\,x^2}.$

20. $\int \frac{dx}{\sqrt{12-3\,x-x^2}}.$

21. $\int \frac{dx}{\sqrt{x^2+6\,x+1}}.$

22. $\int \frac{dz}{z^2-6}.$

23. $\int \frac{d\theta}{4-3\,\theta^2}.$

24. $\int \frac{dx}{x\sqrt{1-\log^2 x}}.$

25. $\int \frac{ds}{\sqrt{2\,as+s^2}}.$

26. $\int \frac{du}{\sqrt{(u+m)^2-n^2}}.$

27. $\int \tan(mx + b)\,dx.$

28. $\int \frac{e^{-x}\,dx}{a + be^{-x}}.$

29. $\int \frac{x^{-1}\,dx}{\log x}.$

30. $\int \frac{x^5\,dx}{\sqrt{a^{12} - x^{12}}}.$

31. $\int \frac{e^x - \sin x + \cos x}{e^x + \sin x + \cos x}\,dx.$

32. $\int \frac{dx}{2x^2 + 2x + 5}.$

33. $\int (2x - 7) \sin(x^2 - 7x)\,dx.$

34. $\int \frac{x - 3}{x\sqrt{x}}\,dx.$ $2\sqrt{x} + \frac{6}{\sqrt{x}} + C$

35. $\int \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\,dx.$

60. Harmonic functions. In this and the following sections of this chapter we shall consider some simple applications to physical problems.

Functions of the type

$$y = a \sin(bx + c)$$
$$y = a \cos(bx + c)$$
$$y = m \sin bx + n \cos bx$$

are called *harmonic functions* of x. The first two forms may readily be reduced to the third, or vice versa. These functions play an important part in many physical problems.

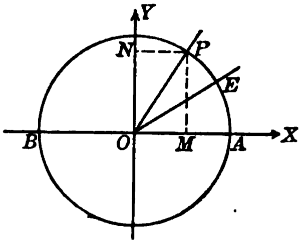

Fig. 36.

(a) *Harmonic motion.* A radius OP, Fig. 36, rotates about the center O with a constant angular speed ω. In consequence of this motion, the point M, which is the projection of P on the horizontal diameter AB, moves from A to B and back, and is said to have *simple harmonic motion.* Taking O as origin, let $OM = x$, and $OP = a$. If t denotes the time occupied in the rotation from OA to OP, then the angle $AOP = \omega t$. Hence, since

$$OM = OP \cos AOP,$$

$$x = a \cos \omega t. \qquad (1)$$

Differentiating, we get

$$\frac{dx}{dt} = v = -a\omega \sin \omega t. \qquad (2)$$

This expression gives the velocity of M along the diameter AB.

Differentiating again, we have

$$a = \frac{dv}{dt} = - a\omega^2 \cos \omega t = - x\omega^2, \qquad (3)$$

which is the acceleration of the point M. It will be observed that the displacement, velocity, and acceleration of the moving point are all harmonic functions.

(b) *Harmonic electromotive force.* A second example of a harmonic function is found in the electromotive force (e.m.f.) and the current induced in a wire moving in a magnetic field. If a rectangular coil of area a be placed at right angles to a magnetic field, the intensity of which is n lines of force per unit area, the number of lines passing through the coil is na. When the coil is turned about its axis AB through an angle θ, the number of lines N threading through the coil is evidently

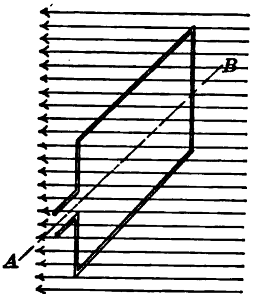

Fig. 37.

$$N = na \cos \theta. \qquad (4)$$

The induced e.m.f. (E) is given by the rate at which the coil cuts the lines of force; that is,

$$E = \frac{dN}{dt}. \qquad (5)$$

We have therefore

$$E = - na \sin \theta \frac{d\theta}{dt}.$$

But $\frac{d\theta}{dt}$ is the angular speed ω of the coil about its axis; hence

$$E = - na\omega \sin \theta. \qquad (6)$$

If the e.m.f. is wholly used to overcome the resistance R of the circuit, the current i flowing is

$$i = \frac{E}{R} = - \frac{na\omega}{R} \sin \theta. \qquad (7)$$

Hence the e.m.f. and current are both harmonic functions of the same angle. If the circuit contains inductance in addition to resistance, the current will be a function of $\theta - \epsilon$. The current lags behind the e.m.f. by the angle ϵ, which is called the angle of lag (see Ex. 6). Since, with constant speed of rotation, $\theta = \omega t$, we may, if we choose, use t as the independent variable.

EXERCISES.

1. Deduce expressions for the displacement, velocity, and acceleration of the point N, Fig. 36, which is the projection of P on the Y-axis.

2. Draw displacement, velocity, and acceleration curves,
(a) with the stroke AB as a base line;
(b) with the values of the angle AOP as abscissas.

3. In Fig. 36, let t be taken as the time reckoned from an initial position OE, making an angle 30° with OA. Deduce equations for displacement, velocity, and acceleration for this condition.

4. Show that the sum of two harmonic currents $i_1 = A\sin(\theta - \epsilon_1)$ and $i_2 = B\sin(\theta - \epsilon_2)$ is equivalent to a single harmonic current.

5. In Ex. 4, take $A = 3$, $B = 4$, $\epsilon_1 = 0$, $\epsilon_2 = 60°$, and draw graphs of the two functions i_1 and i_2 and of the sum $i_1 + i_2$.

6. When the circuit has inductance, the e.m.f. is given by the equation

$$E = Ri + L\frac{di}{dt}.$$

The part Ri is the e.m.f. required to overcome the resistance, and the part $L\dfrac{di}{dt}$ is the e.m.f. used to increase the current, where L is the inductance. Show that if i is a harmonic function, E is also a harmonic function.

Suggestion: Since

$$i = A\sin\omega t,$$

$$\frac{di}{dt} = A\omega\cos\omega t,$$

whence $E = RA\sin\omega t + LA\,\omega\cos\omega t = A\sqrt{R^2 + L^2\omega^2}\sin(\omega t + \epsilon)$,

where $\tan\epsilon = \dfrac{L\omega}{R}$. Hence the current lags behind the e.m.f.

61. The compound interest law. If a function has the general form

$$y = e^{ax},$$

then

$$D_x y = ae^{ax} = ay;$$

that is, the rate of change of the function is proportional to the function itself. Many natural phenomena follow this law, which has been called by Lord Kelvin the "compound interest" law.

The following are illustrations:

(a) *Atmospheric pressure.* It is well known that the pressure of the atmosphere decreases as the distance from the earth's surface increases. Assuming the temperature of the atmosphere to be constant, the rate of change of pressure with the height at any given height is proportional to the pressure at that height; that is,

$$\frac{dp}{dh} = -kp.$$

The negative sign is used because p decreases as h increases. We have then

$$\frac{dp}{p} = -kdh,$$

and integrating, we get

$$\log p = -kh + C.$$

To determine the constant of integration C, let p_0 denote the pressure at the earth's surface where $h = 0$. Then

$$\log p_0 = 0 + C, \text{ or } C = \log p_0.$$

Substituting this value of C and transposing, we have

$$kh = \log p_0 - \log p = \log \frac{p_0}{p},$$

whence

$$\frac{p_0}{p} = e^{kh},$$

or

$$p = p_0 e^{-kh}.$$

This formula gives the pressure at any definite height, when the constant k is known.

(b) *Newton's law of cooling.* A body has a temperature τ, higher than the temperature τ_0 of the surrounding medium. The rate at which it cools is approximately proportional to the difference in temperature $\tau - \tau_0$; that is, approximately,

$$\frac{d\tau}{dt} = -k(\tau - \tau_0),$$

where k is some constant. We have then

$$\frac{d\tau}{\tau - \tau_0} = -kdt,$$

whence by integration,

$$\log(\tau - \tau_0) = -kt + C.$$

Let τ_1 denote the temperature when $t = 0$; then

$$C = \log(\tau_1 - \tau_0),$$

and substituting this value of C, we get

$$\log \frac{\tau_1 - \tau_0}{\tau - \tau_0} = kt,$$

or

$$\frac{\tau_1 - \tau_0}{\tau - \tau_0} = e^{kt}.$$

Hence $$\tau = \tau_0 + (\tau_1 - \tau_0)\, e^{-kt},$$
from which τ can be found for any time interval.

(c) *Inversion of sugar.* Cane sugar in solution is decomposed into other substances through the presence of acids. The rate at which the process takes place is proportional to the mass of sugar still unchanged. Thus, if s is the original mass of sugar and x is the mass inverted, the rate of inversion at this stage of the process is proportional to the unchanged mass $s - x$. The equation expressing this law is

$$\frac{dx}{dt} = k\,(s - x).$$

Integrating, we get

$$-\log (s - x) = kt + C.$$

To determine the constant of integration C, we make use of the fact that $x = 0$ when $t = 0$, whence $C = -\log s$. Substituting this value of C in the original equation, we have

$$\log \frac{s}{s - x} = kt,$$

or $$\frac{s}{s - x} = e^{kt}.$$

Solving for x, we get

$$x = s\,(1 - e^{-kt}).$$

(d) *Electric currents.* The general equation expressing the relation between the electromotive force (e.m.f.) and current in a circuit is

$$E = Ri + L\frac{di}{dt}\cdot \quad \text{(See Ex. 6, p. 134.)}$$

Case 1. A constant e.m.f. E is connected to the circuit, and the value of the growing current t seconds after the e.m.f. is connected is required.

From the general equation, we have

$$\frac{di}{dt} = \frac{E}{L} - \frac{Ri}{L},$$

whence $$\frac{di}{E - Ri} = \frac{dt}{L}.$$

Integrating, we get

$$-\log (E - Ri) = \frac{Rt}{L} + C.$$

Since $i = 0$ when $t = 0$, we have $C = -\log E$. Substituting, we get

$$\log \frac{E}{E - Ri} = \frac{Rt}{L},$$

whence $$\frac{E}{E - Ri} = e^{\frac{Rt}{L}}.$$

Therefore $$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right).$$

Case 2. Let the e.m.f. be removed, and at the instant of removal, when $t = 0$, let the value of the current be i_0; the value of the current at any succeeding instant is required.

Since in this case $E = 0$, we have

$$0 = Ri + L\frac{di}{dt},$$

whence

$$\frac{di}{Ri} = -\frac{dt}{L}.$$

Integrating, we obtain

$$\log i = -\frac{R}{L}t + C.$$

Since $i = i_0$ when $t = 0$, we have $C = \log i_0$,

whence

$$\log\frac{i}{i_0} = -\frac{R}{L}t,$$

or

$$i = i_0 e^{-\frac{Rt}{L}}.$$

EXERCISES.

1. Find the amount of P_0 dollars at the end of t years at r % compound interest, supposing the interest to be added to the principal at the end of each $1/n$ th of a year. Show that the limiting form of the expression for the amount as n is indefinitely increased is

$$P = P_0 e^{\frac{rt}{100}}.$$

Show that the same expression is obtained directly if it is assumed that the interest is added continuously.

Take $P_0 = \$1000$, $t = 2$ years, $r = 6\%$; find the amount on the assumption that the interest is added (a) yearly, (b) semi-annually, (c) quarterly, (d) monthly, (e) daily, (f) continuously.

2. The retarding effect of fluid friction on a rapidly rotating disk is proportional to the angular speed, that is,

$$\frac{d\omega}{dt} = -k\omega.$$

If ω_0 denotes the initial angular speed, show that

$$\omega = \omega_0 e^{-kt}.$$

MISCELLANEOUS EXERCISES.

1. Derive the formulas for the derivatives of $\cos u$, $\tan u$, and $\sec u$ directly from the defining equation

$$\underset{\Delta u \doteq 0}{L}\frac{\Delta y}{\Delta u} = \underset{\Delta u \doteq 0}{L}\frac{f(u + \Delta u) - f(u)}{\Delta u}$$

and the theorem of Art. 21.

2. Obtain the derivative of $\cot u$ from the relation $\cot u = \frac{1}{\tan u}$; also the derivative of $\sec u$ from $\sec u = \sqrt{1 + \tan^2 u}$.

3. Write out formulas for the derivatives of the inverse circular functions (see Art. 53) for the case in which $u = \frac{x}{a}$.

4. Discuss the following curves by means of their derivatives. Find where they cross the X-axis, the angles at which they cross, and the points at which their tangents are parallel to the X-axis:

(a) $y = \sin\left(x - \frac{\pi}{4}\right)$. (c) $y = \sin x + \cos x$.

(b) $y = \sin 2x$. (d) $y = \log x$.

5. When $\theta = 36°$, what is the rate of increase of (a) $\sin\theta$; (b) $\cos\theta$; (c) $\tan\theta$ compared with the rate of increase of θ?

6. An angle θ is determined by drawing a perpendicular to AB, as BC, Fig. 38, and finding the ratio $\frac{BC}{AB}$. Suppose $AB = 5$ and $BC = 2.8$. What error, expressed in minutes of arc, will be made in the calculated value of the angle if an error of 0.03 is made in the measurement of BC?

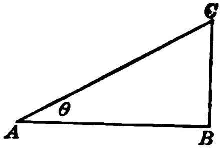

Fig. 38.

7. The cycloid is given by the equations

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta).$$

Derive an expression for the slope of the tangent, and find the value of this slope when $a = 3$, $\theta = \frac{2}{3}\pi$. For what value of θ is the tangent parallel to the X-axis?

$0 + k\pi$

8. Differentiate the following:

(a) $\sqrt{a^2 - x^2} - a \log \frac{a + \sqrt{a^2 - x^2}}{x}$.

(b) $\frac{x - 3a}{2}\sqrt{2ax + x^2} + \frac{3}{2}a^2 \log(x + a + \sqrt{2ax + x^2})$.

(c) $\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$.

(d) $\frac{1}{x} - \frac{1}{3x^3} + \arctan x$. (e) $\log(e^x + e^{-x})$. (f) $\tan a^{\frac{1}{x}}$.

9. Find the polar subtangents and subnormals of the following curves:

(a) $\rho = a(1 + \cos\theta)$. (c) $\rho = a^2 \sin^2\theta$.

(b) $\rho^2 = e^{a\theta}$. (d) $\rho = \sec^2\frac{\theta}{2}$.

10. Find expressions for $\tan\psi$ (Art. 28) for the curves of Ex. 9.

11. If a point moves in a logarithmic spiral $\rho = e^{a\theta}$, show that the components of its velocity along and perpendicular to the radius vector have a constant ratio.

12. A point moves in the curve $\rho = a\,(1 - \cos\theta)$ with a speed of m units. Find general expressions for the velocity components along and perpendicular to the radius vector.

13. The area of a triangle is determined by the measurement of two sides and the included angle. Find the relative error in the area due to an error ΔC in the measurement of the included angle C.

14. Integrate the following:

(a) $\displaystyle\int \frac{dx}{e^x + e^{-x}}$. (c) $\displaystyle\int \frac{dx}{\sinh x}$. (e) $\displaystyle\int \frac{3\,x\,dx}{x^4 + 5}$.

(b) $\displaystyle\int \frac{e^{2x}\,dx}{(e^x + 1)^{\frac{1}{2}}}$. (d) $\displaystyle\int \frac{x^4\,dx}{x^2 + 1}$. (f) $\displaystyle\int \frac{5\,dx}{x\sqrt{9\,x^2 - 4}}$.

(g) $\displaystyle\int \frac{dx}{(x - a)\sqrt{(x - a)^2 - b^2}}$.

15. Write out integration formulas corresponding to the derivatives (18)–(20), Art. 56, taking $u = \dfrac{x}{a}$.

16. The following formulas give approximately the relation between the pressure and temperature of saturated steam:

(a) $\log p = A + \dfrac{B}{T} + \dfrac{C}{T^2}$. $(T = \tau + 461)$

(b) $p = ab^{\frac{\tau}{c + \tau}}$.

(c) $\log p = a + b\alpha^{\theta} - c\beta^{\theta}$. $(\theta = \tau - 32)$

From each formula derive an expression for the rate of change of the pressure with the temperature.

17. When a body falls vertically in a resisting medium, as air, the acceleration is given by the relation

$$a = g - kv^2,$$

in which g is the acceleration due to gravity, and k is a constant. Derive an expression for the velocity of the body, and show that this velocity approaches a definite limiting value $\sqrt{\dfrac{g}{k}}$ as t increases without limit.

18. Derive an expression for the velocity of a point moving in a straight line if the acceleration is

$$a = C - kv.$$

19. A point moves in accordance with the equation

$$s = ae^{-\lambda t} \cos 2\,\pi\,(bt + c).$$

Derive an expression for the velocity.

CHAPTER VIII

SUCCESSIVE DIFFERENTIATION AND INTEGRATION

62. Definition of the nth derivative. The derivative of a function is, in general, a function of the same independent variable as the original function, and it may itself have a derivative if it fulfills the necessary conditions. The differentiation of the original function produces the first derivative; the differentiation of this first derivative gives the second derivative; the result of differentiating this second derivative is the third derivative, and so on. If the process is repeated n times, the final result is the nth derivative of the function with respect to its independent variable.

As an example, consider the function

$$f(x) = x^3 - 2x + 5.$$

The first derivative is

$$3x^2 - 2,$$

which is also a function of x. Differentiating this, the second derivative is $6x$, which is again a function of x. The third derivative is 6, a constant, and the fourth and each successive derivative is zero.

If $y = f(x)$ is the original function, the successive derivatives are denoted by the symbols:

$$D_x y,\ D_x^2 y,\ D_x^3 y,\ \ldots\ D_x^n y,$$

or by

$$f'(x), f''(x), f'''(x), \ldots, f^n(x).$$

A general formula for the nth derivative may be found for certain functions as shown by the following examples.

Ex. 1. Let $y = x^p$, where p is a positive integer.

Then

$$\begin{aligned} D_x y &= px^{p-1}, \\ D_x^2 y &= p(p-1)x^{p-2}, \\ D_x^3 y &= p(p-1)(p-2)x^{p-3}, \\ &\ldots\ldots\ldots\ldots \\ D_x^n y &= p(p-1)(p-2)\ \ldots\ (p-n+1)x^{p-n}. \end{aligned}$$

For $n = p$, $x^{p-n} = 1$ and $D_x^p(x^p) = p\,!$; hence for $n > p$, the derivative is zero.

Ex. 2. Let
$$y = e^{ax},$$
then we have
$$\begin{aligned} D_x y &= ae^{ax}, \\ D_x^2 y &= a^2e^{ax}, \\ D_x^3 y &= a^3e^{ax}, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ D_x^n y &= a^n e^{ax}. \end{aligned}$$

63. Successive differentials. It has been shown that if
$$y = f(x), \tag{1}$$
then
$$dy = f'(x)\,dx, \tag{2}$$
where dy, dx represent the rates of change with respect to some common variable t. As x is the independent variable, we may choose any value we please for the rate of change of x, that is, for dx. For the sake of convenience and simplicity, we shall assume dx to be constant; in other words, we assume that the rate of change of the independent variable is constant. This assumption enables us to obtain a very simple relation between the successive derivatives and differentials.

We have from (2) (noting that dy stands for D_ty),
$$\begin{aligned} D_t(D_ty) = D_t^2y &= D_t[f'(x)\,dx] \\ &= D_xf'(x) \cdot D_tx \cdot dx, \qquad \text{(Art. 21)} \end{aligned}$$
or
$$d^2y = f''(x)\,(dx)^2. \tag{3}$$
The second differential of y we denote by d^2y. In general, the nth differential is denoted by d^ny. In the second member dx appears twice as a factor but not as a higher differential; this follows from the fact that dx is constant. We shall ordinarily denote the nth power of dx by dx^n rather than by $(dx)^n$, but this symbol must not be confused with $d(x^n)$. We may now deduce the following relations by successive differentiation.

$$d^3y = f'''(x)\,dx^3, \qquad \frac{d^3y}{dx^3} = f'''(x) = D_x^3y. \tag{4}$$

$$d^4y = f^{\text{IV}}(x)\,dx^4, \qquad \frac{d^4y}{dx^4} = f^{\text{IV}}(x) = D_x^4y. \tag{5}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$d^ny = f^n(x)\,dx^n, \qquad \frac{d^ny}{dx^n} = f^n(x) = D_x^ny. \tag{6}$$

These relations hold so long as x is the independent variable, and we can therefore assign arbitrary values to dx. If x ceases to be the independent variable, and its rate of change depends upon that of a third variable, these relations no longer hold and must be replaced by other and more complicated ones.

Ex. Given $y = 4x^3 + 6x + 7$.
By differentiation, we have

$$\begin{aligned} dy &= 12x^2\,dx + 6\,dx \\ &= (12x^2 + 6)\,dx, & \frac{dy}{dx} &= D_xy = 12x^2 + 6, \\ d^2y &= 24x\,dx^2, & \frac{d^2y}{dx^2} &= D_x^2y = 24x, \\ d^3y &= 24\,dx^3, & \frac{d^3y}{dx^3} &= 24. \end{aligned}$$

64. Successive differentiation of implicit functions. The following example illustrates the method of procedure in finding successive derivatives of an implicit function.

Ex. Given $f(x, y) = x^2y + 5y - 3x = 0$; find $\dfrac{d^2y}{dx^2}$.
Differentiating with respect to x, we obtain

$$2xy + x^2D_xy + 5D_xy - 3 = 0, \tag{1}$$

whence

$$D_xy = \frac{dy}{dx} = \frac{3 - 2xy}{x^2 + 5}. \tag{2}$$

A second differentiation with respect to x gives

$$\begin{aligned} D_x^2y &= \frac{(x^2 + 5)\,D_x(3 - 2xy) - (3 - 2xy)\,D_x(x^2 + 5)}{(x^2 + 5)^2} \\ &= \frac{-(x^2 + 5)(2y + 2x\,D_xy) - (3 - 2xy)\,2x}{(x^2 + 5)^2}. \end{aligned} \tag{3}$$

Substituting in (3) the expression for D_xy given in (2), we obtain after reduction

$$D_x^2y = \frac{d^2y}{dx^2} = \frac{6x^2y - 12x - 10y}{(x^2 + 5)^2}. \tag{4}$$

The same result can be obtained by using differentials throughout. Thus from the original equation, we obtain

$$2xy\,dx + x^2dy + 5\,dy - 3\,dx = 0. \tag{5}$$

A second differentiation (remembering that dx is constant) gives

$$2y\,dx^2 + 2x\,dy\,dx + 2x\,dx\,dy + x^2d^2y + 5\,d^2y = 0. \tag{6}$$

From (5) we obtain

$$\frac{dy}{dx} = -\frac{2xy - 3}{x^2 + 5}, \tag{7}$$

and from (6)

$$2y + 4x\frac{dy}{dx} + (x^2 + 5)\frac{d^2y}{dx^2} = 0.$$

Substituting the value of $\frac{dy}{dx}$ from (6) and solving, we have finally

$$\frac{d^2y}{dx^2} = \frac{6x^2y - 12x - 10y}{(x^2 + 5)^2}. \tag{8}$$

Observe that each term of (5) contains a first differential, while each term of (6) contains either a second differential or the product of two first differentials.

65. Geometrical and physical interpretations of the second derivative. As has been shown, the first derivative D_xy $\left(\text{or } \frac{dy}{dx}\right)$ gives the rate of change of y with respect to x, and expresses geometrically the slope of the curve which is the graph of $y = f(x)$. Evidently, the second derivative D_x^2y $\left(\text{or } \frac{d^2y}{dx^2}\right)$ gives the rate of change of the slope compared with the rate of change of the abscissa x. Take, for example, the function $y = mx + b$, which is represented by a straight line. The slope is constant, and therefore its rate of change is zero. This is shown by the derivatives; for $D_xy = m$ and $D_x^2y = 0$.

In the case of a moving point, the speed v is the derivative D_ts, while the tangential acceleration a is D_tv. If now D_ts be substituted for v, we have

$$a = D_t(D_ts) = D_t^2s,$$

or

$$a = \frac{d^2s}{dt^2};$$

that is, *the tangential acceleration is the second time-derivative of the space s.*

In the case of rotation about a fixed axis, the angular speed is $D_t\theta = \frac{d\theta}{dt} = \omega$, and the angular acceleration is

$$\alpha = D_t\omega = D_t(D_t\theta) = D_t^2\theta = \frac{d^2\theta}{dt^2};$$

that is, *the angular acceleration is the second time-derivative of the angle swept over.*

EXERCISES.

1. Find the second derivatives of the following:

(a) $y = x^3 - 3x^2 + 5$.
(b) $y = x^n$.
(c) $y = e^{ax} \sin x$.
(d) $y = x \operatorname{arc} \cos x$.
(e) $y = \sec x$.
(f) $y = \operatorname{arc} \tan \theta$.

2. Find $D_x{}^3 y$ when:

(a) $y = x^2 \log \frac{x}{a}$;
(b) $y = \tan x$;
(c) $y = x \tan x$;
(d) $y = \sin^3 x \cos x$.

3. Find the first five derivatives of the following:

(a) $y = 3x^4 - 7x^3 + 5x - 7$.
(b) $y = \log (x + 2)$.
(c) $y = x^4 \log x^2$.

4. Find the *n*th derivatives of

(a) $y = a^x$;
(b) $y = \frac{1}{x}$;
(c) $y = \log x$.

5. Find $\frac{d^2y}{dx^2}$ from (a) the equation of the parabola $y^2 = 4px$; (b) the equation of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

6. Find $\frac{d^2y}{dx^2}$ when

(a) $x^3y + 3x^2y^2 + xy^3 = 0$;
(b) $y^2 - 2axy + x^2 - c = 0$;
(c) $y = \cos (x + 5)$.

7. If $pv^n = C$, show that $\frac{d^2p}{dv^2} = n(n + 1)\frac{p}{v^2}$.

8. Find the values of $\frac{d^3r}{d\theta^3}$ for the following:

(a) $r = \cosh \frac{\theta}{a}$;
(b) $r = \log \sin \theta$;
(c) $r = \frac{\theta}{1 - \cos \theta}$;
(d) $r = ae^{n\theta}$.

9. If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

10. If $y = \log [x + \sqrt{a^2 + x^2}]$, show that $\frac{d^2y}{dx^2} + \frac{x}{a^2 + x^2}\frac{dy}{dx} = 0$.

66. Successive integration. As the inverse of successive differentiation, we have the process of successive integration. Starting with a function $y = f(x)$, considered as an *n*th derived function, a single integration gives a new function, the anti-derivative or integral. The integration of this second function gives a second anti-derivative or integral, and so on. The result of n integrations is the nth integral of the given function.

For example, let

$$f(x) = 5x^2.$$

Then $D_x^{-1}(5x^2) = \frac{5}{3}x^3 + C_1$, is the first anti-derivative,

$D_x^{-1}(\frac{5}{3}x^3 + C_1) = \frac{5}{12}x^4 + C_1x + C_2$, is the second anti-derivative;

$D_x^{-1}(\frac{5}{12}x^4 + C_1x + C_2) = \frac{1}{12}x^5 + \frac{1}{2}C_1x^2 + C_2x + C_3$, is the third anti-derivative, etc.

It will be observed that an anti-derivative contains the number of arbitrary constants indicated by its order; thus the third anti-derivative has three, the fourth, four, and so on.

The successive integrals of $y = f(x)$ may be denoted by the symbols:

$$D_x^{-1}y,\ D_x^{-2}y,\ \ldots,\ D_x^{-n}y,$$

where

$$D_x^{-2}y = D_x^{-1}(D_x^{-1}(y)), \text{ etc.}$$

These are all functions of x, say $f_1(x), f_2(x), \ldots, f_n(x)$, respectively.

It is customary, however, to use the notation of differentials. Since $f_1(x)$ is the first integral of $f(x)$, we have

$$f_1(x) = \int f(x)\,dx. \tag{1}$$

Likewise,

$$f_2(x) = \int f_1(x)\,dx, \tag{2}$$

$$f_3(x) = \int f_2(x)\,dx, \tag{3}$$

and so on. Substituting in (3) the value of $f_2(x)$ given by (2), we have

$$f_3(x) = \int\left(\int f_1(x)\,dx\right)dx, \tag{4}$$

and substituting in this result the value of $f_1(x)$ from (1), we get

$$f_3(x) = \int\left[\int\left(\int f(x)\,dx\right)dx\right]dx. \tag{5}$$

This expression is more conveniently written in the form

$$f_3(x) = \int\int\int f(x)\,dx^3. \tag{6}$$

Likewise we may write

$$f_4(x) = \int\int\int\int f(x)\,dx^4, \tag{7}$$

and in general

$$f_n(x) = \int\int\int\int\ \cdots\ \int\int f(x)\,dx^n. \tag{8}$$

Ex. 1. Find the successive integrals of $y = e^{ax}$.

$$f_1(x) = \int f(x)\,dx = \int e^{ax}\,dx = \frac{1}{a}e^{ax} + C_1,$$

$$f_2(x) = \int\left(\frac{1}{a}e^{ax} + C_1\right)dx = \frac{1}{a^2}e^{ax} + C_1x + C_2,$$

$$f_3(x) = \int\left(\frac{1}{a^2}e^{ax} + C_1x + C_2\right)dx = \frac{1}{a^3}e^{ax} + \frac{1}{2}C_1x^2 + C_2x + C_3,$$

. . . .

$$f_n(x) = \frac{1}{a^n}e^{ax} + k_1x^{n-1} + k_2x^{n-2} + \ldots + k_{n-1}x + k_n,$$

in which $k_1, k_2, \ldots, k_n$ are constants of integration.

Ex. 2. A body falls with a constant acceleration $g = 32.2$ ft./sec². If it starts from rest, through what distance will it fall in 10 seconds?

We have here
$$\frac{d^2s}{dt^2} = 32.2, \tag{9}$$

$$\frac{ds}{dt} = \int\frac{d^2s}{dt^2}\,dt = \int 32.2\,dt = 32.2\,t + C_1, \tag{10}$$

$$s = \int\frac{ds}{dt}\,dt = \int(32.2\,t + C_1)\,dt = 16.1\,t^2 + C_1t + C_2. \tag{11}$$

We may now determine the constants C_1, C_2 from the initial conditions of the problem. Since the body falls from rest, we have for $t = 0$,

$$s = 0,\quad \frac{ds}{dt} = 0.$$

Hence for $t = 0$, we have from (10) and (11)

$$C_1 = 0,\quad C_2 = 0.$$

Therefore, for the time t, the space passed over is given by

$$s = 16.1\,t^2.$$

For $t = 10$ seconds, $s = 1610$ feet.

EXERCISES.

1. If $y = \cos mx$, find $D_x^{-1}y$, $D_x^{-2}y$, $D_x^{-3}y$, and $D_x^{-4}y$.

2. Find $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$, when $f(x) = x^3 + 1$.

3. Find three successive integrals of $y = \frac{1}{x^3}$.

4. Find a curve which passes through the points (a, b) and (m, n) and for which the second derivative $\frac{d^2y}{dx^2}$ at any point is k times the abscissa at that point.

5. A point has an acceleration expressed by the equation

$$a = -r\omega^2\cos\omega t,$$

where r and ω are constants. Derive expressions for the velocity and distance traveled.

We have
$$\frac{d^2s}{dt^2} = -r\omega^2\cos\omega t.$$

It may occur that both $f'(x)$ and $f''(x)$ vanish for the same value a. In this case the last method of testing the given function for maxima and minima fails, and we must apply the first method. Theorem I must also be applied when $f'(x)$ changes sign by passing through infinity, as x passes through $x = a$, or by having a finite discontinuity, as in Fig. 39, (a) and (b) respectively.

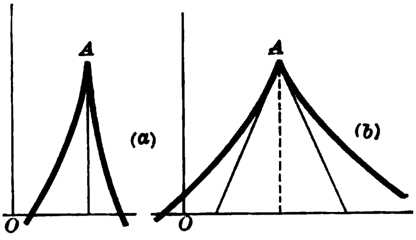

Fig. 39.

The special cases where higher derivatives vanish simultaneously with $f'(x)$ will be considered in a later section, where a second proof purely analytic in its nature will be given.

Ex. 1. Examine the function $(x + 2)^2 (x - 1)^3$ for maximum and minimum values.

$$f(x) = (x + 2)^2 (x - 1)^3,$$

$$f'(x) = 2 (x + 2) (x - 1)^3 + 3 (x + 2)^2 (x - 1)^2$$

$$= 5 (x + 2) (x - 1)^2 (x + \tfrac{4}{5}).$$

From $$(x + 2) (x - 1)^2 (5 x + 4) = 0,$$

we have the roots $x = -2, +1, -\frac{4}{5}$. These are critical values to be examined.

We may test the value -2 by examining the original function; thus

$$f(-2) = 0,$$

$$f(-2 + h) = h^2 (-2 + h - 1)^3 < 0,$$

$$f(-2 - h) = h^2 (-2 - h - 1)^3 < 0.$$

Therefore, since the function is greater for $x = -2$ than for values of x just preceding or following it, it follows that for $x = -2$, the function has a maximum. The value $x = 1$ we shall likewise examine by substituting $x = 1$, $x = 1 + h$, and $x = 1 - h$ in the original function.

$$f(1) = 0,\ f(1 + h) = (1 + h + 2)^2 h^3 > 0,$$

$$f(1 - h) = -(1 - h + 2)^2 h^3 < 0.$$

For this value, the function has neither a maximum nor a minimum, although $f'(x) = 0$. For the critical value $-\frac{4}{5}$, we may examine the derivative $f'(x)$.

For $x = -\frac{4}{5} - h$, $f'(x)$ is negative.

For $x = -\frac{4}{5} + h$, $f'(x)$ is positive.

Since $f'(x)$ changes from negative to positive as x passes through the value $-\frac{4}{3}$, the function has a minimum for this value of x.

The graph of this function is shown in Fig. 40.

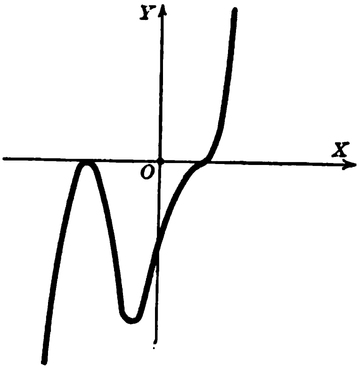

Fig. 40.

Ex. 2. Examine $x^3 - 4x^2 + 5x - 2$ for maxima and minima.

$f(x) = x^3 - 4x^2 + 5x - 2,$

$f'(x) = 3x^2 - 8x + 5 = (3x - 5)(x - 1),$

$f''(x) = 6x - 8.$

Putting $f'(x) = 0$, the critical values are seen to be $x = \frac{5}{3}$, $x = 1$. For $x = \frac{5}{3}$, $f''(x) = 6 \times \frac{5}{3} - 8 = 2$, and for $x = 1$, $f''(x) = -2$; hence, the function has a minimum for $x = \frac{5}{3}$ and a maximum for $x = 1$.

Ex. 3. Let
$$f(x) = \frac{x}{\log x}.$$

Then
$$f'(x) = \frac{\log x - 1}{(\log x)^2}.$$

Hence for $f'(x) = 0$, $\log x = 1$ or $x = e$ gives the critical value. For $x < e$, $\log x < 1$, and $f'(x)$ is negative, while for $x > e$, $\log x > 1$ and $f'(x)$ is positive; hence, $f(e)$ is a minimum value of $f(x)$.

Ex. 4. Examine $(x - 4)^{\frac{2}{3}} + 5$ for maxima and minima.

Here
$$f(x) = (x - 4)^{\frac{2}{3}} + 5,$$

whence
$$f'(x) = \frac{2}{3(x - 4)^{\frac{1}{3}}}.$$

For $x = 4$, $f'(x) = \infty$; hence $x = 4$ is a critical value. For $x > 4$, $f'(x)$ is positive, while for $x < 4$, $f'(x)$ is negative. Hence as x increases, $f'(x)$ changes from negative to positive, and from Theorem I, $f(x)$ has a minimum at $x = 4$. See Fig. 41.

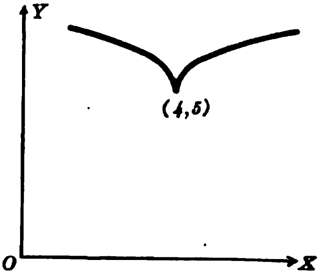

Fig. 41.

EXERCISES.

Examine the following functions for maximum and minimum values:

1. $\frac{x^3}{3} - \frac{x^2}{2}$.
2. $(x - 1)(x - 2)^2$.
3. $2x^3 - 15x^2 + 36x + 6$.
4. $x^3 - 3x^2 - 9x + 5$.
5. $\frac{(4 - x)^3}{4 - 2x}$.
6. $a_1 e^{mx} + a_2 e^{-mx}$.
7. $2\cos\theta + \sin^2\theta$.
8. x^x.

Hence
$$v = \frac{ds}{dt} = \int \frac{d^2s}{dt^2}\,dt = -\,r\omega^2 \int \cos \omega t\,dt$$
$$= -\,r\omega \sin \omega t + C',$$

and
$$s = \int \frac{ds}{dt}\,dt = -\,r\omega \int \sin \omega t\,dt + \int C'dt$$
$$= r \cos \omega t + C't + C'',$$

which is the law of simple harmonic motion.

6. In Ex. 5, determine the constants of integration from the following initial conditions: when $t = 0$, $s = r$ and $v = 0$.

67. Maxima and minima. In Chapter IV, we discussed the relation of the derivative to increasing and decreasing functions and to turning-points of a curve. At a turning-point, the function has the largest or the smallest value of all values in that neighborhood. We say that it has a maximum or a minimum value at such a point. A maximum value of a function may therefore be defined as follows:

A function $f(x)$ is said to have a *maximum* for $x = a$, if $f(a)$ is greater than the values of $f(x)$ for x just preceding and just following $x = a$. Expressed symbolically $f(x)$ has a maximum for $x = a$, if

$$f(a) > f(a \pm h),$$

where h is positive and takes all values in the immediate neighborhood of zero.

Likewise, a function $f(x)$ is said to have a *minimum* for $x = a$, if $f(a)$ is less than the values of $f(x)$ for x just preceding and just following $x = a$; in other words, if

$$f(a) < f(a \pm h),$$

for all values of h in the neighborhood of zero.

A function may have several maxima and several minima in any given interval. We shall limit the present discussion to continuous functions having only a finite number of maxima and minima in a finite interval.

The analytic conditions for a maximum or a minimum are easily deduced when the function has a derivative. If $f(a)$ is a maximum, $f(x)$ must increase with x just before $x = a$ and decrease as x increases just beyond. As shown in Art. 24, the derived function $f'(x)$ must then be positive for values of x just preceding $x = a$ and negative for values just following. For similar reasons, in case $f(a)$ is a minimum, $f'(x)$ must be negative for values of x just preceding $x = a$ and positive for values just following. We may combine these statements in the following theorem:

THEOREM I. *$f(a)$ is a maximum or a minimum of the function $f(x)$, according as $f'(x)$ changes from positive to negative or from negative to positive values as x, for increasing values, passes through the value $x = a$. If $f'(x)$ does not change sign as x passes through $x=a$, then $f(x)$ has neither a maximum nor a minimum at $x=a$.*

To apply this test, substitute $(a - h)$ and $(a + h)$ for x in $f'(x)$. In case $f(a)$ is a maximum, $f'(a - h)$ remains positive and $f'(a + h)$ negative however small the value of h is taken; if $f(a)$ is a minimum, then $f'(a - h)$ remains negative and $f'(a + h)$ positive for arbitrarily small values of h.

If the maximum or minimum occurs at an ordinary point of the curve, the tangent at the point is parallel to the X-axis, and consequently the first derivative for this value of x must vanish; that is, $f'(a) = 0$. We may determine whether $f'(x)$ passes from positive to negative values or vice versa by examining the character of *its* derivative in the neighborhood of $x = a$. In case of a maximum, $f'(x)$ is first positive, then zero at $x = a$, and finally negative for values following a. In other words, as $f(x)$ passes through the maximum value, $f'(x)$ decreases. Hence, by Art. 24, $f''(x)$ must be either negative or zero for $x = a$. On the other hand, as $f(x)$ passes through a minimum, $f'(x)$ is first negative, then zero, and finally positive; that is, $f'(x)$ is an increasing function at the point $x = a$. Hence $f''(x)$ must be either positive or zero for $x = a$. For the case in which $f'(a)$ becomes zero while $f''(a)$ remains different from zero, we may therefore express the condition for a maximum or a minimum as follows:

THEOREM II. *If $f'(a) = 0$, then $f(a)$ is a maximum or a minimum according as $f''(a)$ is negative or positive.*

This theorem affords an easy method of determining the maximum or the minimum points of a function. We equate the first derivative to zero and solve for the real values of the variable. If upon substituting these values for the variable in the second derivative we obtain a negative number, we have a maximum; if we obtain a positive number, we have a minimum.

9. $\frac{a^2}{x} + \frac{b^2}{a-x}$.

10. $x - \tan x$. max. $x = \pi + 2k\pi$ [illegible]

11. $x\sqrt{ax+b}$. [illegible]

12. $x - e^x$. max. $x = 0, y = -1$

13. $(x-2)^3 + 6$.

14. $\frac{2x}{1+x^2}$.

68. Applications of maxima and minima. The theory of maxima and minima has many important applications in geometry, mechanics, and physics, a few of which are given in the following problems. In some of these problems the function whose maximum or minimum value is required is given; in others, however, the function must be found from the conditions stated in the problem. Frequently the function derived will contain more than one variable. In such cases, the conditions of the problem will furnish certain relations between the variables by means of which all but one can be eliminated. The function of the single variable thus obtained will be treated according to the methods given in the preceding section. A careful study of the following illustrative examples will give a grasp of the proper method of attack.

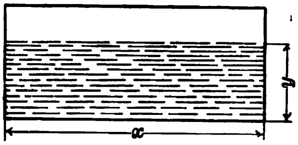

Fig. 42.

Ex. 1. A rectangular channel carrying a given volume of water is to be so proportioned as to have a minimum wetted perimeter (i.e., surface in contact with water). Determine the proportions of the channel.

Let x = width of bottom, and y = height of water. The volume of water is proportional to the cross section; since this volume is to be constant, we have

$$xy = C. \tag{1}$$

The wetted perimeter p is given by the equation

$$p = x + 2y. \tag{2}$$

This is the function whose minimum value is required. It contains two variables x and y, but one may be eliminated by means of the first relation; thus,

$$y = \frac{C}{x},$$

whence

$$p = x + 2\frac{C}{x}. \tag{3}$$

Differentiating (3), we get

$$\frac{dp}{dx} = 1 - \frac{2C}{x^2}. \tag{4}$$

In order that p shall be a maximum or minimum, we must have

$$1 - \frac{2C}{x^2} = 0,$$

whence $x = \sqrt{2C},$

and $y = \frac{1}{2}\sqrt{2C};$

that is, $x = 2y.$ (5)

To determine whether this ratio between bottom and side gives a maximum or minimum, we observe that $\frac{dp}{dx}$ is negative for $x^2 < 2C$ and positive for $x^2 > 2C$. This shows that p is a minimum for $x = 2y$.

Ex. 2. The deflection of a rectangular beam of a fixed length under a given load varies inversely as the product of the breadth and the cube of the depth. From a log a inches in diameter, a beam is to be cut of such dimensions as to make the deflection a minimum.

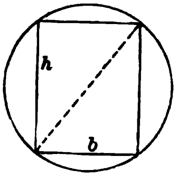

Fig. 43.

Denoting by z the deflection, and by b and h the breadth and depth, respectively, the equation

$$z = \frac{k}{bh^3} \tag{6}$$

expresses the law stated above.

There are two variables in this function, but one of them may be eliminated by means of a second equation derived from the geometry of the figure, viz.:

$$b^2 + h^2 = a^2. \tag{7}$$

Combining (6) and (7), we have

$$z = \frac{k}{h^3\sqrt{a^2 - h^2}}. \tag{8}$$

By differentiation of (8), we obtain

$$\frac{dz}{dh} = \frac{k(4h^2 - 3a^2)}{h^4(a^2 - h^2)^{\frac{3}{2}}}. \tag{9}$$

This derivative takes the value zero when $4h^2 - 3a^2 = 0$, that is, when

$$h = \frac{a}{2}\sqrt{3}. \tag{10}$$

The corresponding value of b is, from (7), $\frac{a}{2}$. The student may show that for these values z is a minimum, not a maximum.

EXERCISES.

1. Divide a number m into two parts such that their product is a maximum.

2. Divide a number a into two parts such that the sum of their squares shall be a minimum.

3. The range of a projectile is given by the expression

$$R = \frac{v_0^2 \sin 2\phi}{g},$$

in which v_0 denotes the initial velocity and ϕ the angle which v_0 makes with the horizontal. For what angle ϕ is the maximum range obtained? What is the maximum range?

4. The efficiency of a screw as a mechanical device is given by the formula

$$E = \frac{h(1 - h\mu)}{h + \mu},$$

where the constant μ is the coefficient of friction, and h is the tangent of the pitch angle of the screw. Find the value of h for which the efficiency is a maximum. $-\mu + \sqrt{\mu^2+1}$

5. It is desired to make an open-top box of greatest possible volume from a square piece of tin whose side is a, by cutting out equal squares from the corners and then folding up the tin to form the sides. What should be the length of the side of the square cut out? $\frac{a}{6}$

6. A roofer wishes to make an open trapezoidal gutter of maximum capacity whose bottom and sides are each 4 inches wide and whose sides have the same slope. What should be the width across the top?

7. Determine the most economical proportions for a cylindrical tank with flat heads; that is, find the ratio of the length to the diameter.

8. A submarine cable consists of a core of copper wires with a covering made of nonconducting material. If x denotes the ratio of the radius of the core to the thickness of the covering, it is known that the speed of signaling varies as $x^2 \log (1/x)$. For what value of x is the greatest speed attained? 1

9. A high vertical wall is to be braced by a beam which must pass over a parallel wall a feet high, and b feet distant from the other. Find the length of the shortest beam that can be used for the purpose.

10. Determine the right circular cone of minimum volume that can be circumscribed about a given sphere.

11. Given a triangle one of whose vertices lies at the center of a circle of radius a. If two sides of the triangle are radii of the circle, show that $\frac{1}{2} a^2$ is the maximum area of the triangle.

12. Find the dimensions of the following inscribed figures that will give maximum area:

(a) Rectangle in a circle of radius r.

(b) Rectangle in ellipse of semiaxes a and b.

(c) Isosceles triangle in circle of radius a.

13. Find the dimensions of the following inscribed solids in order that the volume shall be a maximum:

(a) Cylinder in right circular cone.

(b) Cone in a sphere of radius a.

(c) Cone in a paraboloid of revolution; vertex of cone at intersection of axis of paraboloid with base.

14. Find the dimensions of a cone of maximum volume (a) with given slant-height; (b) with given total surface S.

15. Find the dimensions of a conical tent that for a given capacity V will require the least material.

16. The expression $\dfrac{2a(v-b)^2}{v^3}$ determines the sign of the derivative of van der Waals' equation (Art. 24, Ex. 2). For what value of v is this a maximum?

17. The power P developed by a Pelton water wheel is proportional to the speed v of the wheel buckets and also to the relative velocity $c - v$ with which the jet issuing from the nozzle with speed c strikes the buckets; that is,

$$P = kv\,(c - v),$$

where k is a constant. With a given value of the jet speed c determine the bucket speed v that will give maximum power.

18. The horsepower transmitted by a hemp rope is given by the formula

$$H = \frac{v}{825}\left(T - \frac{wv^2}{g}\right). \qquad \text{(Art. 4, Ex. 6.)}$$

Find the rope speed that gives maximum horsepower. Find the value of this maximum H when $T = 420$, $w = 0.72$.

19. The weight of hot gas passing up a chimney in a given time is given by the formula

$$W = \frac{k\sqrt{.96\,T_1 - T_2}}{T_1},$$

where T_1 and T_2 denote respectively the absolute temperatures of the gas and the outside air. Find the ratio $\dfrac{T_1}{T_2}$ for which W is a maximum.

20. The velocity of a wave in deep water is given by the expression

$$v = \left(\frac{\lambda}{a} + \frac{a}{\lambda}\right)^{\frac{1}{2}},$$

in which a is a constant and λ denotes the wave length. For what wave length is the velocity a minimum?

21. The strength of a rectangular beam varies as the breadth and the square of the depth. Find the dimensions of the beam of maximum strength that can be cut from a log 14 inches in diameter.

22. Find the beam of greatest stiffness that can be cut from a log 12 inches in diameter, knowing that the stiffness varies as the breadth and the cube of the depth.

23. A body of weight W is dragged along a horizontal plane by means of a force P whose line of action makes an angle θ with the plane. The magnitude of the force is given by the equation

$$P = \frac{\mu W}{\mu \sin\theta + \cos\theta},$$

in which μ denotes the coefficient of friction. Show that the pull is least when $\theta = \text{arc}\tan\mu$.

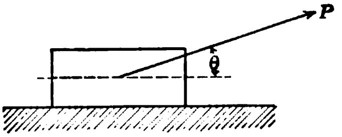

Fig. 44.

24. The turning effect of a ship's rudder is proportional to $\cos\theta\sin^2\theta$, where θ denotes the angle between the rudder and the ship's keel. Find the value of θ that gives maximum turning effect.

25. In a compound air compressor the theoretical work of compression is

$$W = p_a v_a \frac{n}{n-1}\left[\left(\frac{p}{p_a}\right)^{\frac{n-1}{n}} + \left(\frac{p_1}{p}\right)^{\frac{n-1}{n}} - 2\right],$$

in which p_a, p, and p_1 are respectively atmospheric pressure, the pressure in the intermediate receiver, and the final pressure. The constant n is the exponent of the compression curve $pv^n = C$. Determine the relation between p_a, p, and p_1 that the work may be a minimum. Remember that p_a and p_1 are fixed, but p may vary.

69. Concavity of a curve. A curve is said to be concave upward at any point if an arc of the curve containing the point lies above the tangent to the curve at the point. It is said to be concave downward if the tangent lies above the arc. Thus the curve shown in Fig. 45 is concave downward between the points a and b, and it is concave upward from b to c.

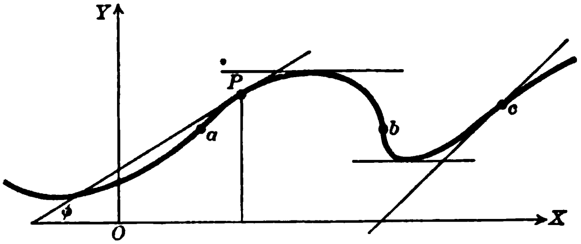

Fig. 45.

The condition for concavity upward or downward can be expressed in terms of the derivatives of the function represented by the curve. As we have seen,

$$\frac{dy}{dx} = \tan\phi. \qquad \text{(Art. 24.)}$$

When the arc of the curve lies below the tangent, as, for example, between a and b, $\tan\phi$, and consequently $\frac{dy}{dx}$, decreases as x increases. On the other hand, when the arc lies above the tangent, $\frac{dy}{dx}$ increases with x. We may therefore say:

A given curve is concave upward or downward according as the derivative $\frac{dy}{dx}$ is an increasing or a decreasing function.

When $\frac{dy}{dx}$ is decreasing, its derivative, namely, $\frac{d^2y}{dx^2}$, must be either negative or zero; if it is increasing, then the second derivative is either positive or zero. The values for which $\frac{d^2y}{dx^2}$ is zero correspond to points on the curve where the curve crosses the tangent. Such cases will be discussed in the next article. A consideration of the values of $\frac{d^2y}{dx^2}$ different from zero enables us to state the following useful criterion:

A given curve is concave upward if the second derivative is positive, and concave downward if the second derivative is negative.

Ex. 1. Given the curve $y = \sin x$.

The first derivative, $\frac{dy}{dx} = \cos x$, is a decreasing function from $x = 0$ to $x = \pi$, and an increasing function from $x = \pi$ to $x = 2\pi$. Hence for values of x between 0 and π, the curve is concave downward, while for $\pi < x < 2\pi$, the curve is concave upward.

Ex. 2. Given the curve $y = x^3 - x^2 + 6$, investigate its concavity.

The second derivative, $\frac{d^2y}{dx^2} = 6x - 2$, is positive for values of $x > \frac{1}{3}$, and is negative for values of $x < \frac{1}{3}$. Hence to the left of $x = \frac{1}{3}$ the curve is concave downward, and to the right of that point it is concave upward.

EXERCISES.

1. Test the following curves for concavity upward or downward:

(a) $y = x^3 - 2x^2 + 5$. (b) $y = \sec x$. (c) $y = \frac{x(1-x)}{1+x}$.

2. Test the concavity of the following curves:

(a) $y = a\sqrt{x-a}$; (b) $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$.

3. Show that a curve is concave or convex toward the X-axis according as $y\frac{d^2y}{dx^2}$ is negative or positive.

4. Show that the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is always concave toward the X-axis.

5. Test the curves $y = e^x$ and $y = \log x$ for concavity.

6. Test the expansion curves given by the general equation $p^m v^n = C$ for concavity.

70. Points of inflexion. In the preceding article we considered the conditions under which a curve can be concave upward or concave downward. We now turn our attention to points at which the curve ceases to be concave upward and becomes concave downward, or vice versa. Such points are called *points of inflexion* of the given curve. The curve shown in Fig. 45 has points of inflexion at a, b, c. It is evident that at such points the curve intersects its tangent.

The above definition suggests the analytic condition for a point of inflexion. As has been shown, so long as the curve is concave upward the first derivative increases with the independent variable, and when it is concave downward the derivative decreases as the variable increases. It follows then that, as the independent variable passes through a value for which the curve has a point of inflexion, the first derivative of the function passes from an increasing to a decreasing function, or vice versa. This is precisely the condition that the derived function shall have a maximum or minimum, and to determine the points of inflexion of a curve, we need only to examine the derived function for maxima and minima. It is therefore a necessary and sufficient condition for a point of inflexion that the second derivative shall change sign as the independent variable passes through the critical value. This change occurs when the second derivative passes through zero or through infinity; hence the coördinates of the points of inflexion of the curve whose equation is

$$y = f(x)$$

may in general be found by solving the equations

$$f''(x) = 0,\ f''(x) = \infty,$$

and determining whether $f''(x)$ changes sign as x passes through the values thus obtained.

Ex. Given the curve whose equation is $y = 5x^3 + 6x - 7$.

We have

$$f''(x) = \frac{d^2y}{dx^2} = 30x,$$

which vanishes for $x = 0$. Moreover, $f''(x)$ changes sign as x passes from negative to positive values. Hence the curve has a point of inflexion at $x = 0$.

EXERCISES.

1. Given a continuous curve. Draw a tangent to this curve, and through some fixed point draw a straight line which shall so change as to remain parallel to the tangent as the point of tangency changes. Show that as the point of tangency passes through a point of inflexion, the line through the fixed point changes the direction of its rotation. For this reason, the tangent at a point of inflexion is sometimes called a stationary tangent. Express in terms of the derivatives of the given function the condition for a stationary tangent.

2. Test the following curves for points of inflexion:

(a) $y = 3x^3 + 19x^2 + 24x + 7.$ (b) $y = 5x^4 + 7x^3 + 6x + 9.$

(c) $(y - 2)^3 = x + 4.$

3. Find the equation of a curve which has a point of inflexion at the point (0, 3) such that the inflexional tangent makes an angle of 45° with the X-axis. How many such curves can be found? [illegible]

4. Test the following curves for points of inflexion:

(a) $y = \tan x.$ (b) $y = xe-x^2.$ (c) $y = \sinh x = \frac{1}{2}(e^x - e^{-x}).$

5. Determine the points of inflexion of the isothermal of van der Waals' equation for $T = 300$. (See Art. 24, Ex. 2.)

6. Show that the curve $y = \dfrac{1 - x}{1 + x^2}$ has three points of inflexion and that these points lie in a straight line.

71. Curve tracing. It is frequently desirable to determine the general form of the graph of a given function. The direct method of tracing a curve is to determine simultaneous values of x and y from the given equation $y = f(x)$ and plot the points thus found. However, by a careful study of the derivatives of the function, the labor of this direct method can be largely avoided, and the general course of the graph, which is all that is required, can be found. In tracing a curve the student should proceed somewhat as follows:

1. Examine the equation for symmetry. If only even powers of x appear, the curve is symmetrical with respect to the Y-axis; if only even powers of y appear, it is symmetrical with respect to the X-axis; while if the equation remains unchanged when x and y are replaced by $-x$ and $-y$ respectively, the curve is symmetrical with respect to the origin.
2. Find the points in which the curve crosses the X- and Y- axes.
3. Find the *finite* values of x (or of y) for which y (or x) becomes infinite.

4. Find the slope of the curve by means of the first derivative and note the turning-points.

5. By an examination of the first or second derivative determine whether the curve is concave upward or downward, and find the points of inflexion.

Having determined these characteristics of the curve, it is easy in most cases to sketch the graph. If a minute study of the form of the curve at any particular point is desired, it is well to transform the equation so that the point in question becomes the origin; then we need only consider the nature of the given curve in the neighborhood of the origin. In determining the nature of the curve in the neighborhood of the origin, the terms of the lowest degree in x and y are in general most important, since for small values of these variables the terms of higher order are small in comparison. Likewise the nature of the curve for very large values of the variables is determined by considering only the terms of highest order in the variables.

In the following examples we shall consider a few of the simpler curves, which are frequently referred to in subsequent chapters. In Chapter XVI we shall develop further principles that will be of service in tracing more complicated curves.

Ex. 1. *The curves* $y = ax^n$. It is assumed that n is positive and either an integer or a fraction. For $n = 2$ and for $n = \frac{1}{2}$, the curves are ordinary parabolas. If $n = 3$ we have the cubical parabola $y = ax^3$. An examination of this equation discloses the following: (a) The curve passes through the origin. (b) For positive values of x, y is positive, and for negative values of x, y is negative; hence the curve lies wholly in the first and third quadrants, and is symmetrical with respect to the origin. (c) $\frac{dy}{dx} = 3\,ax^2$; hence the slope is always positive and increases as $|\,x\,|$ increases. At the origin the slope is zero. (d) $\frac{d^2y}{dx^2} = 6\,ax$, hence to the right of the Y-axis the curve is concave upward, and to the left of the Y-axis it is concave downward. (e) For $x = 0$, $\frac{d^2y}{dx^2} = 0$; hence the origin is a point of inflexion. From these conclusions it is evident that the curve has the general form shown in Fig. 46.

If $n = \frac{3}{2}$, the equation becomes $y^2 = a^2x^3$, and the curve is the semi-cubical parabola. From the equation we get

$$\frac{dy}{dx} = \pm\,\frac{3}{2}a\sqrt{x}, \qquad \frac{d^2y}{dx^2} = \pm\,\frac{3}{4}\,\frac{a}{\sqrt{x}}.$$

It is readily seen that the curve is symmetrical with respect to the X-axis and lies wholly to the right of the Y-axis. At the origin the slope is zero, and the two branches of the curve have the X-axis as a common tangent. There is no point of inflexion. Hence the curve has the general form shown in Fig. 47.

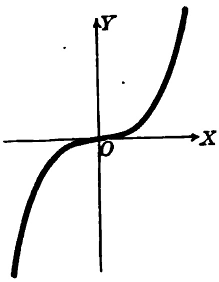

Fig. 46.

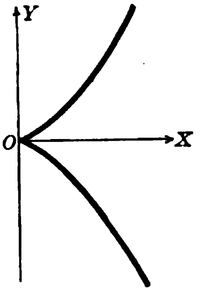

Fig. 47.

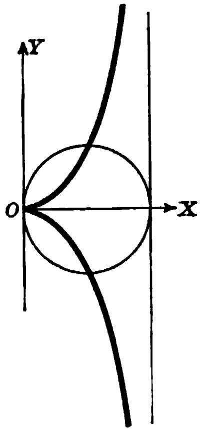

Fig. 48.

Ex. 2. *The cissoid* $y^2 = \dfrac{x^3}{2a - x}\cdot$ The curve is symmetrical with respect to the X-axis. For $x = 2a$, y becomes infinite; and for $x > 2a$ or $x < 0$, y is imaginary; hence the curve lies wholly between the Y-axis and the line $x = 2a$, and the two branches approach tangency with the latter line at an infinite distance. For $x = 0$, $\dfrac{dy}{dx} = 0$, showing that the X-axis is a tangent to both branches at the origin. An investigation of the second derivative shows that there is no point of inflexion and that both branches are convex towards the X-axis. The curve has therefore the general form shown in Fig. 48.

Ex. 3. *The witch* $y = \dfrac{8a^3}{x^2 + 4a^2}\cdot$ The curve is symmetrical with respect to the Y-axis. y has a maximum value $2a$ for $x = 0$, and there are points of inflexion at $x = \pm 2a\sqrt{3}$. The two branches approach the X-axis at an infinite distance. The curve is shown in Fig. 49.

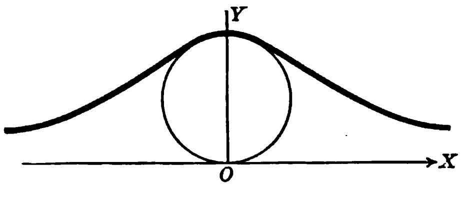

Fig. 49.

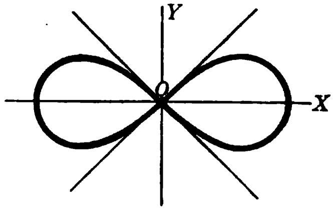

Fig. 50.

Ex. 4. *The lemniscate* $\rho^2 = a^2 \cos 2\theta$,
or

$$(x^2 + y^2)^2 = a^2 (x^2 - y^2).$$

The curve is symmetrical with respect to both axes, and crosses the X-axis at $x = \pm a$. For values of θ from $\frac{1}{4}\pi$ to $\frac{3}{4}\pi$ and from $\frac{5}{4}\pi$ to $\frac{7}{4}\pi$, ρ is imaginary. The curve has the form shown in Fig. 50.

Ex. 5. *The catenary* $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ is the curve assumed by a flexible cord of uniform weight suspended from two fixed points. The curve is symmetrical about the Y-axis, and cuts the Y-axis at a distance a above the origin. The second derivative is positive for all values of x, hence the curve is concave upwards and has no point of inflexion. See Fig. 51.

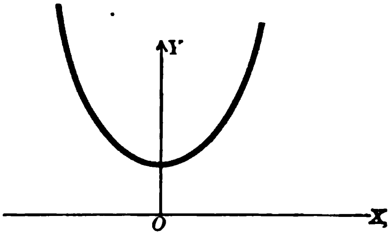

Fig. 51.

Ex. 6. The *cycloid* is the curve described by a point on the circumference of a circle which rolls on a straight line. Let a denote the radius of the circle, and

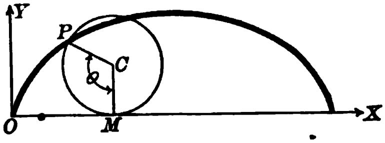

Fig. 52.

θ the angle PCM, Fig. 52, subtended by the arc PM ($= OM$). Then we have for the coördinates of P

$$x = a(\theta - \sin\theta),$$

$$y = a(1 - \cos\theta).$$

From these equations we readily obtain

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}, \quad \frac{d^2y}{dx^2} = -\frac{1}{a(1 - \cos\theta)^2}.$$

The curve has a turning-point at $x = \pi a$, and since $\frac{d^2y}{dx^2}$ is always negative, the curve is everywhere concave downward.

Ex. 7. The *astroid* or hypocycloid of four cusps $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is a curve described by a point on the circumference of a circle of radius $\frac{1}{4}a$ rolling within the circumference of a circle of radius a. See Fig. 53. The student may find the derivatives and by their aid study the curve for slope, concavity, etc.

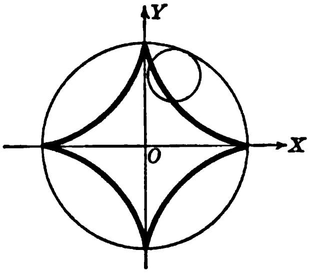

Fig. 53.

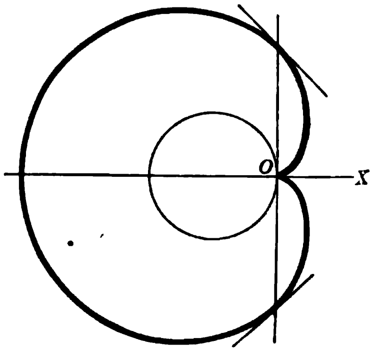

Fig. 54.

Ex. 8. The *cardioid* $\rho = 2a(1 - \cos\theta)$. The curve is closed, and ρ is finite for all values of θ. Since $\cos\theta = \cos(-\theta)$, the curve is symmetrical with respect to the initial line OX. From the given equation, we get

$$\tan\psi = \rho\frac{d\theta}{d\rho} = \frac{1 - \cos\theta}{\sin\theta} = \tan\tfrac{1}{2}\theta, \qquad \text{(Art. 28.)}$$

whence

$$\psi = \tfrac{1}{2}\theta.$$

For $\theta = 0$, $\psi = 0$, that is, OX is a tangent at the origin; for $\theta = \frac{1}{2}\pi$, $\psi = \frac{1}{4}\pi$, and for $\theta = \pi$, $\psi = \frac{1}{2}\pi$. For the limits of ρ we have $\rho = 0$ when $\theta = 0$, $\rho = 4a$ when $\theta = \pi$. See Fig. 54.

EXERCISES.

The student should trace the following curves carefully and preserve the graphs for future reference.

1. The group of curves $x^m y = C$ (m positive). These are the curves which represent the laws of expanding gases.

2. The curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. Show that this curve is the ordinary parabola.

3. The logarithmic spiral $\rho = e^{a\theta}$.

4. The spiral of Archimedes, $\rho = a\theta$.

5. The curve $\rho = a\sec^2\frac{\theta}{2}$.

6. The parabolic spiral $\rho^2 = a^2\theta$.

7. The probability curve $y = e^{-x^2}$. Find turning-points and points of inflexion.

T.P. $x = 0$; P.I. $x = \mp\sqrt{\frac{1}{2}}$

8. The lituus $\rho^2\theta = a^2$.

9. The curves $\rho = a \sin n\theta$, $\rho = a \cos n\theta$. (Give n values 1, 2, 3, 4.)

10. Show that the cardioid is described by a point on the circumference of a circle of radius a which rolls upon a fixed circle of the same radius.

72. Curvature. If a point moves along a plane curve as EF Fig. 55, the direction of motion at any point is the direction of the tangent to the curve. The direction of the tangent continually changes, and a comparison of this change of direction with the distance traversed by the point leads us to the idea of curvature. Thus, as the point moves from P to Q, the tangent turns through the angle $\Delta\phi$, and we say the curvature of the arc PQ is large or small according as the angle $\Delta\phi$ is large or small.

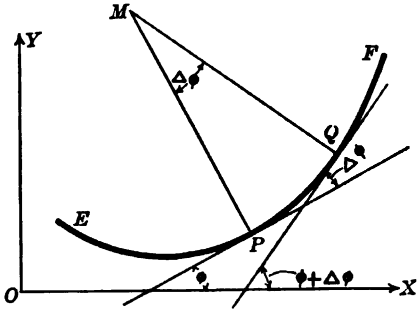

Fig. 55.

Denoting by Δs the length of the arc PQ, the ratio $\frac{\Delta\phi}{\Delta s}$ is defined as the *mean curvature* of the arc PQ. Provided the curvature is constant, this quotient is the curvature at all points between P and Q; but if the curvature is not constant, then the *curvature at P* is defined as the limit

$$\underset{\Delta s \doteq 0}{L} \frac{\Delta\phi}{\Delta s} = D_s\phi = \frac{d\phi}{ds}. \tag{1}$$

The curvature of the curve $y = f(x)$ at the point (x_1, y_1) may be expressed in terms of the derivatives $f'(x)$ and $f''(x)$. We have the fundamental relation

$$\tan\phi = f'(x),\ \phi = \text{arc} \tan f'(x),$$

whence by differentiation

$$D_x\phi = \frac{D_x f'(x)}{1 + (f'(x))^2} = \frac{f''(x)}{1 + (f'(x))^2}.$$

Furthermore we have

$$D_x s = \sqrt{1 + (f'(x))^2}.^* \qquad \text{(See Art. 47.)}$$

From these equations we obtain by division

$$\frac{d\phi}{ds} = \frac{D_x \phi}{D_x s} = \frac{f''(x)}{[1 + (f'(x))^2]^{\frac{3}{2}}}. \tag{2}$$

At the point (x_1, y_1), therefore, the curvature is

$$\frac{f''(x_1)}{[1 + (f'(x_1))^2]^{\frac{3}{2}}}. \tag{3}$$

73. Radius of curvature. Center of curvature. Suppose that PQ, Fig. 55, is an arc of a circle. Let normals be drawn at P and Q intersecting at M; then $MP = MQ$ is the radius of the arc, and angle $PMQ = \Delta\phi$. Denoting the radius by r, we have

$$r\Delta\phi = \Delta s,$$

whence

$$\frac{\Delta\phi}{\Delta s} = \frac{1}{r}; \tag{1}$$

that is, the curvature of a circle is constant and is the reciprocal of the radius.

At any point of a curve $y = f(x)$, conceive a circle drawn tangent to the curve, and suppose it to have the same curvature as the curve at that point. This circle is called the *circle of curvature;* its radius, the *radius of curvature;* and its center, the *center of curvature* of the curve at the given point. Since the circle and curve have the same curvature at the point in question, the radius of curvature R must be the reciprocal of this curvature, that is,

$$R = \frac{ds}{d\phi}.$$

In terms of the derivatives involved, the formula for the radius of curvature at the point (x_1, y_1) becomes

$$R = \frac{[1 + (f'(x_1))^2]^{\frac{3}{2}}}{f''(x_1)}. \tag{2}$$

By hypothesis, the curve and the circle of curvature at the given point have a common tangent; hence the radius of curvature has the direction of the normal to the curve.

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[*] A derivation of this relation is given in Art. 88. For the present, the student may make use of Eq. (1), Art. 47.

It is customary in the case of single-valued functions to regard R positive or negative according as the curve is concave upward or concave downward. To establish this convention, it is necessary to consider the positive value only of the radical in the numerator. The sign of R then depends upon that of $f''(x)$, and the required result follows from Art. 70. In many cases it is the numerical value of R alone that is of importance.

As we have seen, $f''(x) = \frac{d^2y}{dx^2}$ changes sign at a point of inflexion; hence R also changes sign at such a point. This property might have been used as the definition of a point of inflexion. Thus a point of inflexion is a point at which the radius of curvature, and consequently the curvature itself, changes sign.

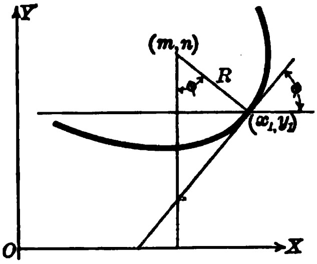

Fig. 56.

The *coördinates of the center of curvature* may be found as follows: If (m, n), Fig. 56, is the center of curvature corresponding to the point (x_1, y_1) of the curve, we have from the figure,

$$x_1 - m = R \sin \phi,$$

$$n - y_1 = R \cos \phi.$$

But $$\tan \phi = f'(x_1),$$

whence $$\cos \phi = \frac{1}{\sqrt{1 + [f'(x_1)]^2}},$$

and $$\sin \phi = \frac{f'(x_1)}{\sqrt{1 + [f'(x_1)]^2}}.$$

Using these expressions for $\sin \phi$ and $\cos \phi$, and the expression for R given by (2), we obtain after reduction

$$\left.\begin{aligned} m &= x_1 - \frac{f'(x_1)[1 + (f'(x_1))^2]}{f''(x_1)}; \\ n &= y_1 + \frac{1 + [f'(x_1)]^2}{f''(x_1)}. \end{aligned}\right\} \qquad (3)$$

Ex. Find the radius of curvature of the equilateral hyperbola $xy = c^2$ at the point (x_1, y_1); also at the point (c, c).

We have $\quad y = \frac{c^2}{x}, \qquad f'(x_1) = -\frac{c^2}{x_1^2}, \qquad f''(x_1) = \frac{2c^2}{x_1^3}.$

Substituting these values in (1), we get

$$R = \frac{\left[1 + \frac{c^4}{x_1^4}\right]^{\frac{3}{2}}}{\frac{2c^2}{x_1^3}} = \frac{(x_1^2 + y_1^2)^{\frac{3}{2}}}{2c^2}.$$

At the point (c, c), therefore,

$$R = \frac{(2c^2)^{\frac{3}{2}}}{2c^2} = c\sqrt{2}.$$

Also at this point $f'(x) = -1$, and $f''(x) = \frac{2}{c}$. Substituting in (3), we find for the coördinates of the center of curvature,

$$m = n = 2c.$$

EXERCISES.

Find the radius of curvature of each of the following curves at the point (x_1, y_1).

1. $y^2 = 2ax$.
2. $ay^2 = x^3 + b$.
3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
4. $y = \log(x + a)$.
5. $y = \log\sec x$.
6. $y = \operatorname{sech} x$.
7. $y = \cosh x$.
8. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
9. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.
10. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
11. $y^2 = x^{\frac{3}{2}} - 5$.
12. $y^2 = \frac{x^3}{2a - x}$.
13. $\sqrt{x} + \sqrt{y} = 2\sqrt{c}$.
14. $e^{\frac{y}{a}} = \sec\frac{x}{a}$.

For the following curves find the radius of curvature and the coördinates of the center of curvature at the points indicated.

15. $y^2 = 10x$, at $x = 5$.
16. $xy = 30$, at $(3, 10)$.
17. $y = \cos x$, at $(0, 1)$.
18. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$, at $(0, b)$.
19. $xy^2 = 4a^2(2a - x)$, at $(2a, 0)$.

74. Radius of curvature, parametric representation. If a curve is given by parametric equations, as

$$x = f(\theta), \quad y = \phi(\theta),$$

we may obtain an expression for the radius of curvature as follows:

We have the general formula

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}, \tag{1}$$

from which by the aid of Art. 21, we obtain

$$\frac{d^2y}{dx^2} = \frac{\dfrac{dx}{d\theta}\dfrac{d^2y}{d\theta^2} - \dfrac{dy}{d\theta}\dfrac{d^2x}{d\theta^2}}{\left(\dfrac{dx}{d\theta}\right)^2} \; \frac{d\theta}{dx}$$

$$= \frac{\dfrac{dx}{d\theta}\dfrac{d^2y}{d\theta^2} - \dfrac{dy}{d\theta}\dfrac{d^2x}{d\theta^2}}{\left(\dfrac{dx}{d\theta}\right)^3}. \tag{2}$$

Substituting these expressions in formula (2), Art 73, we obtain

$$R = \frac{\left[\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\dfrac{dx}{d\theta}\dfrac{d^2y}{d\theta^2} - \dfrac{dy}{d\theta}\dfrac{d^2x}{d\theta^2}}. \tag{3}$$

Written in differential form (3) becomes

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\, d^2y - dy\, d^2x}. \tag{4}$$

Ex. Find the radius of curvature of the cycloid

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta).$$

By successive differentiation, we obtain

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \qquad \frac{dy}{d\theta} = a\sin\theta,$$

$$\frac{d^2x}{d\theta^2} = a\sin\theta, \qquad \frac{d^2y}{d\theta^2} = a\cos\theta.$$

Substituting these values in (3), we find

$$R = \frac{a^3 2^{\frac{3}{2}} (1 - \cos\theta)^{\frac{3}{2}}}{a^2 (\cos\theta - 1)} = - a\sqrt{8}\sqrt{1 - \cos\theta} = - 4\,a \sin\frac{\theta}{2}.$$

If a curve is given in polar coördinates, we have

$$x = \rho \cos\theta, \quad y = \rho \sin\theta.$$

By means of these relations, (3) is reduced to the form

$$R = \frac{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2 + 2\left(\frac{d\rho}{d\theta}\right)^2 - \rho\frac{d^2\rho}{d\theta^2}}. \tag{5}$$

Hence, using the functional symbols for the derivatives, the radius of curvature of the curve $\rho = F(\theta)$ at the point (ρ_1, θ_1) is

$$R = \frac{[\rho_1{}^2 + (F'(\theta_1))^2]^{\frac{3}{2}}}{\rho_1{}^2 + 2\,[F'(\theta_1)]^2 - \rho_1 F''(\theta_1)}. \tag{6}$$

EXERCISES.

Find the radius of curvature of each of the following curves at the point (ρ_1, θ_1).

1. $\rho = a\theta$.
2. $\rho\theta = c$.
3. $\rho = a\,(2\cos\theta - 1)$.
4. $\rho = a\sec^2\frac{\theta}{2}$.
5. $\rho^2 \cos 2\,\theta = a^2$.
6. $\rho = a\,(1 - \cos\theta)$.
7. $\rho = e^{a\theta}$.

8. Find the radius of curvature of the ellipse from the equations

$$x = a\cos\theta,$$
$$y = b\sin\theta.$$

75. Other expressions for R. In rectangular coördinates, equation (2), Art. 73, is the usual formula for R. There are, however, other formulas which are in some cases more convenient. In stating these formulas we shall drop the subscripts that indicate the particular point (x_1, y_1).

Recalling the relation

$$(ds)^2 = (dx)^2 + (dy)^2,$$

from which follows $\frac{ds}{dx} = \pm\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$

we may write the formula for R in rectangular coördinates in the following convenient form:

$$R = \frac{\left(\frac{ds}{dx}\right)^3}{\frac{d^2y}{dx^2}}. \tag{1}$$

In the theory of the flexure of beams, the formula for R is simplified as follows: The axis of the unloaded beam is taken as coincident with the X-axis. Since the bending is slight, the slope of the curve which the axis assumes under a load is very small; hence we may take $f'(x) = 0$, and we have approximately

$$R = \frac{1}{f''(x)}. \tag{2}$$

This approximation is frequently used in other applications, as the curvature of lenses, etc.

Ex. The equation of the axis of a beam of length L under certain conditions of loading is

$$EIy = \frac{w}{48}(3\,L^2x^2 - 2\,x^4).$$

Find the radius of curvature [using (2)] for $x = 0$. Take the following values: $E = 30\,000{,}000$, $I = 210$, $w = 60$, $L = 200$.

MISCELLANEOUS EXERCISES.

1. Find $\frac{d^2y}{dx^2}$ for the following:

(a) $x^3 - ax^2y + y^3 = 0$;
(b) $y = e^x \sin\left(x + \frac{\pi}{4}\right)$;
(c) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$;
(d) $y = \frac{a^2x}{(x-a)^2}$;
(e) $y = x^2 \sin x$.

2. If $y = Ae^{ax} + Be^{-ax}$, show that $\frac{d^2y}{dx^2} - a^2y = 0$.

3. If $s = C_1 \sin kt + C_2 \cos kt$, show that $\frac{d^2s}{dt^2} + k^2s = 0$.

4. If u and v are functions of x, show that

$$D_x^n(uv) = uD_x^nv + nD_x^{n-1}vD_xu + \frac{n(n-1)}{2\,!}D_x^{n-2}vD_x^2u + \ldots + vD_x^nu.$$

Observe that the coefficients and symbolic exponents follow the law of coefficients and exponents in the expansion of a binomial. This is *Leibnitz' theorem.*

5. Using the result of Ex. 4, find

(a) D_x^5y when $y = e^x \cos x$;
(b) D_x^4y when $y = x^3 \log x$.

6. The crank OA, Fig. 57, rotates about O with constant angular speed, and through the rod AB gives a reciprocating motion to the point B. Let $r = OA$ and $L = AB$; then the distance of B from the outer end of its stroke M is

$$s = L + r - r\cos\theta - \sqrt{L^2 - r^2\sin^2\theta}.$$

Find the velocity and acceleration of B for any given value of θ.

Suggestion: Remember that $\dfrac{d\theta}{dt}$ is the constant angular speed ω_0.

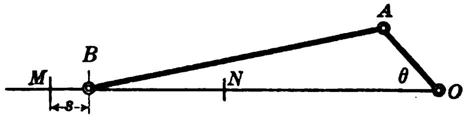

Fig. 57.

7. In Ex. 6 determine the acceleration of B for $\theta = 0, \dfrac{\pi}{2}, \pi$, and on the stroke MN as a base line draw approximately the acceleration curve. Take $L = 4r$.

8. Find the curve whose second derivative $\dfrac{d^2y}{dx^2}$ has the constant value $2a$ and which passes through the points (0, 0) and (2, 6).

9. If $\dfrac{d^2y}{dx^2} = -\dfrac{P}{EI}(L - x)$, find $y = f(x)$, knowing that $y = 0$ and $\dfrac{dy}{dx} = 0$, when $x = 0$.

10. Find three successive integrals of

(a) $x^2 + 3$; (b) e^{mx}; (c) $\sin(kt + \epsilon)$.

11. The weight of gas flowing through an orifice in a given time from a vessel in which the pressure is p_1, into a space in which it is p, is proportional to

$$\left(\frac{p}{p_1}\right)^{\frac{2}{k}} - \left(\frac{p}{p_1}\right)^{\frac{k+1}{k}}.$$

For air, $k = 1.41$. Find the ratio $\dfrac{p}{p_1}$ for which the flow is a maximum.

12. The following formulas give the theoretic efficiency of various types of water and steam turbines. In each, u denotes the peripheral speed of the wheel and is the variable. The other quantities are constants. Find the value of u that makes the efficiency a maximum, and find also the value of the maximum efficiency.

(a) $E = 4\dfrac{u}{c}\left(\cos\alpha - \dfrac{u}{c}\right)$;

(b) $E = ku\,(2c\cos\alpha - u)$;

(c) $E = \dfrac{cu\sqrt{2gh + u^2} - u^2}{gh}$.

13. Examine for maxima and minima and for points of inflexion the curves

(a) $x^3 - axy + b^3 = 0,$ [illegible]

(b) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$ [illegible] $x = -b$

Trace the curves.

14. The intensity of an overhead light varies as the cosine of the angle with the vertical, and inversely as the square of the distance from the light to the illuminated point. What should be the height of an electric light pole to give the strongest illumination at a point 40 feet from the foot of the pole?

15. Derive the formula for the radius of curvature in polar coördinates directly from the defining equation $R = \frac{ds}{d\phi}$ by the aid of the relations

$$\frac{ds}{d\phi} = \frac{D_\theta s}{D_\theta \phi}, \text{ and } \phi = \theta + \psi. \qquad \text{(See Art. 28.)}$$

16. A point which moves in a curve with a speed v has an acceleration component directed towards the center of curvature, the magnitude of which is $\frac{v^2}{R}$ (R is the radius of curvature).

(a) Find an expression for this acceleration component when the point moves on an ellipse. (b) Find this component when the point moves through the position $x = 6$, $y = 6$ on the parabola $y^2 = 6x$ with a speed of 6 units.

17. Find the radius of curvature of the curves given by the equations:

(a) $y = a \sin^3 \theta, \; x = a \cos^3 \theta;$

(b) $y = b\,(2 \sin \theta - \sin 2\theta), \; x = b\,(2 \cos \theta + \cos 2\theta).$

18. If x and y are both functions of an independent variable t, show that the successive derivatives of y with respect to x are as follows:

$$f'(x) = \frac{dy}{dx},$$

$$f''(x) = \frac{d^2y\,dx - d^2x\,dy}{dx^3},$$

$$f'''(x) = \frac{d^3y\,dx^2 - 3\,d^2y\,d^2x\,dx + 3\,dy\,(d^2x)^2 - dy\,dx\,d^3x}{dx^5}.$$

Suggestion: Proceed as in Art. 63, but consider dx a variable.

CHAPTER IX

DEFINITE INTEGRALS

76. Definition of a definite integral. Suppose we have given a function $f(x)$ which is continuous and single-valued within a given interval (a, b), and suppose further that

$$\int f(x)\, dx = D_x^{-1} f(x) = \phi(x) + C. \tag{1}$$

The difference of the values of the function $\phi(x) + C$ for any two values of the independent variable is called a *definite integral.* The values of the independent variable substituted are called the *limits of integration.* We shall denote a definite integral symbolically by writing the two limits of integration at the extremities of the sign of integration. Thus, if $x = a$ and $x = b$ are the limits of integration in the definite integral formed from (1), we indicate that fact by writing this definite integral as follows:

$$\int_a^b f(x)\, dx = [D_x^{-1} f(x)]_a^b = \phi(b) - \phi(a). \tag{2}$$

This is read: "The definite integral of $f(x)$ between the limits $x = a$ and $x = b$" or "from a to b." It is to be noted that in the definite integral, the constant of integration disappears. This follows from the definition. For, we have

$$[\phi(x) + C]_a^b = (\phi(b) + C) - (\phi(a) + C) = \phi(b) - \varphi(a). \tag{3}$$

To distinguish definite integrals from those previously discussed, we shall call the latter *indefinite integrals.*

EXERCISES.

Verify the following:

1. $\int_1^5 4\, x^3\, dx = 624.$
2. $\int_1^2 (3\, x - x^3)\, dx = \frac{3}{4}.$
3. $\int_0^\beta \sin\theta\, d\theta = 1 - \cos\beta.$
4. $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{6}.$

5. $\int_{v_1}^{v_2}\frac{dv}{v} = \log\frac{v_2}{v_1}$.

6. $\int_1^4\frac{dy}{y} = \log 4$.

7. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\tan\phi\, d\phi = 0$.

8. $\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\, d\theta = \frac{1}{2}$.

Evaluate the following definite integrals:

9. $\int_1^3\frac{x\,dx}{3+x^2}$. $\frac{1}{2}$ [illegible]

10. $\int_2^4 x^{-\frac{3}{2}}dx$. $-(1+\sqrt{2})$

11. $\int_0^{\frac{\pi}{4}}\cos 2\theta\, d\theta$. $\frac{1}{2}$

12. $\int_0^{\frac{\pi}{2}}\sin x\, dx$. 1

13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\, d\theta$. 2

14. $\int_0^r\left(\frac{r^2}{a^2-x^2}\right)^{\frac{1}{2}}dx, r < a$.

15. $\int_1^a\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)x^{-\frac{1}{3}}\,dx$.

16. $\int_0^{\pi}\sin\theta\, d\theta$. 2

17. $\int_0^{\frac{\pi}{2}}\frac{\cos\theta\, d\theta}{1+\sin^2\theta}$. [illegible]

18. $\int_0^1\sqrt{1+9x^4}\cdot x^3\,dx$. $\frac{1}{18}(10\sqrt{10}-1)$

19. Show from known formulas of physics that the definite integral $\int_{t_1}^{t_2} gt\, dt$ gives the space traversed by a falling body in the time interval $t_2 - t_1$.

20. Plot the curve $v = f(t) = gt$ (see Fig. 2), and show from geometry that the area between this curve, the t-axis, and the ordinates at t_1 and t_2 is numerically equal to $\int_{t_1}^{t_2} gt\, dt$.

21. Evaluate $\int_1^4 (3x^2 - 4)\,x\,dx$ and $\int_4^1 (3x^2-4)\,x\,dx$. What is the relation between the two integrals? State a general law covering the interchange of the limits of integration.

22. Show that $\int_2^5 (x^2 - 3x)\,dx + \int_5^8 (x^2-3x)\,dx = \int_2^8 (x^2-3x)\,dx$. What general law does this result suggest?

23. Show that the general theorems for indefinite integrals (Art. 39) apply also to definite integrals.

77. Elementary properties of definite integrals. The general properties of integrals already developed apply equally well to definite integrals. Thus, the definite integral of a constant times a function is equal to that constant times the definite integral of the function; the sum of a finite number of definite integrals is equal to the definite integral of the sum of the functions, etc. In addition to these, there are certain properties which apply to definite integrals alone. Among the more elementary of these are the following:

THEOREM I. *Interchanging the limits of integration changes the sign of the integral;* that is,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

From the definition of a definite integral, we have

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a), \tag{1}$$

and

$$\int_b^a f(x)\,dx = \phi(a) - \phi(b). \tag{2}$$

Hence,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \tag{3}$$

which was to be demonstrated.

THEOREM II. *If c lies between a and b, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

For, we have

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a), \tag{4}$$

$$\int_a^c f(x)\,dx = \phi(c) - \phi(a), \tag{5}$$

$$\int_c^b f(x)\,dx = \phi(b) - \phi(c). \tag{6}$$

Adding (5) and (6), we obtain

$$\begin{aligned}\int_a^c f(x)dx + \int_c^b f(x)\,dx &= \phi(c) - \phi(a) + \phi(b) - \phi(c)\\ &= \phi(b) - \phi(a)\\ &= \int_a^b f(x)\,dx, \end{aligned} \tag{7}$$

which establishes the theorem. Evidently the theorem may be extended to include any finite number of values between *a* and *b*.

EXERCISES.

1. Evaluate the integral $\int (3x^2 + 5)\,dx$, using successively the limits of integration, (1, 3), (3, 4), (4, 6), and (1, 6), and verify Theorem II by the result.

2. By evaluating the integrals, show that

$$\int_0^{\frac{\pi}{2}} \sin\theta\,d\theta + \int_{\frac{\pi}{2}}^{\pi} \sin\theta\,d\theta = \int_0^{\pi} \sin\theta\,d\theta.$$

3. Show that $\int_a^b \phi(u)\,du = \int_a^b \phi(x)\,dx$. State this result in the form of a theorem.

78. Change of limits. In the process of finding an integral, it is sometimes convenient to change the variable (see Art. 59). In this case, we may by properly changing the limits of integration obtain the definite integral without a second substitution. Suppose we have the integral $\int_a^b f(x)\,dx$ and make the substitution $z = F(x)$. For $x = a$, we have $z = F(a)$, and for $x = b$, $z = F(b)$; hence the substitution of $F(a)$ and $F(b)$ as limits of integration in the transformed integral must lead to the same result as the substitution of a and b in the original integral.

Ex. Find $\int_0^{\frac{1}{2}} \frac{\arcsin x\,dx}{\sqrt{1-x^2}}$, assuming $z = \arcsin x$.

We have $dz = \frac{dx}{\sqrt{1-x^2}}$, whence $\frac{\arcsin x\,dx}{\sqrt{1-x^2}} = z\,dz$.

For $x = 0$, $z = \arcsin 0 = 0$, the lower limit, and for $x = \frac{1}{2}$, $z = \arcsin\frac{1}{2} = \frac{\pi}{6}$, the upper limit.

Hence
$$\int_0^{\frac{1}{2}} \frac{\arcsin x\,dx}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{6}} z\,dz = \frac{\pi^2}{72}.$$

EXERCISES.

1. $\int_0^2 \frac{x\,dx}{1+x^4}$. Let $z = x^2$.

2. $\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{a^2-x^2}}$. Let $z = \frac{x}{a}$.

3. $\int_0^{\frac{\pi}{2}} \sin^2 x\cos x\,dx$. Let $z = \sin x$.

4. $\int_0^2 e^{x^2} x\,dx.$ Let $z = x^2$.

5. $\int_0^1 \frac{dx}{e^x + e^{-x}}.$ Let $z = e^x$.

6. $\int_0^{\frac{\pi}{4}} \frac{\sin\theta\,d\theta}{\cos^3\theta}.$ Let $z = \tan\theta$.

7. $\int_1^2 \frac{(\log^2 x)\,dx}{x}.$ Let $z = \log x$.

8. $\int_0^a \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}.$ Let $z = \arc\tan\frac{x}{a}$.

79. Areas determined by summation. In the present article, we shall show how we may find by the summation method the area bounded by a curve, the axis of the independent variable, and two ordinates corresponding to two given values of the independent variable.

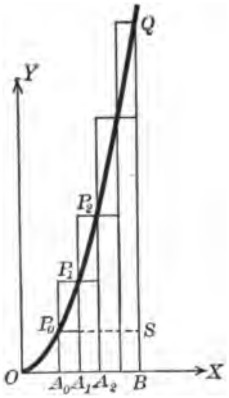

Fig. 58.

For example, let $y = x^2$ be the equation of the given curve (Fig. 58), and let us undertake to find the area bounded by this curve, the X-axis, and the two ordinates $x = 1$, $x = 3$.

Divide that portion of the X-axis between $x = 1$ and $x = 3$ into n equal parts, and denote each by Δx. At each of the points of subdivision, $A_1, A_2, \ldots A_{n-1}$, erect the corresponding ordinates. The required area A_0P_0QB lies between the sum of the outer rectangles A_0P_1, A_1P_2, A_2P_3, etc., and the sum of the inner rectangles P_0A_1, P_1A_2, P_2A_3, etc. This fact may be expressed briefly thus:

$$\sum A_iP_{i+1} > \text{area } A_0P_0QB > \sum P_iA_{i+1}\,. \tag{1}$$

The difference between the sum of the outer and the sum of the inner rectangles is the sum of the small rectangles P_0P_1, P_1P_2, P_2P_3, etc. This difference is therefore given by

$$\sum P_iP_{i+1} = (BQ - A_0P_0)\cdot\Delta x = SQ\cdot\Delta x\,. \tag{2}$$

As n increases indefinitely, or what is the same thing, as $\Delta x \doteq 0$, $SQ\cdot\Delta x$ also approaches zero. Hence the two summations in the inequality (1) have the same limit, and we may write (Art. 11, Th. VI)

$$\underset{\Delta x \doteq 0}{L}\sum A_iP_{i+1} = \text{area } A_0P_0QB = \underset{\Delta x \doteq 0}{L}\sum P_iA_{i+1}\,. \tag{3}$$

We may now calculate the area A_0P_0QB as follows:

The abscissas of the points A_1, A_2, A_3, etc., are $1 + \Delta x, 1 + 2\,\Delta x$, etc., respectively, and the altitudes of the inner rectangles are $1, (1 + \Delta x)^2, (1 + 2\,\Delta x)^2$, etc. The sum of the areas of the inner rectangles is therefore

$$\Delta x + (1 + \Delta x)^2\,\Delta x + (1 + 2\,\Delta x)^2\,\Delta x + (1 + 3\,\Delta x)^2\,\Delta x + \dots \text{ to } n \text{ terms.} \quad (4)$$

As $\Delta x \doteq 0$, the value of n increases, and hence in the limit we have

$$\begin{aligned} \text{area } A_0P_0QB &= \underset{\Delta x \doteq 0}{L} \{\Delta x + (1 + \Delta x)^2\,\Delta x + (1 + 2\,\Delta x)^2\,\Delta x + \dots \\ &\qquad + (1 + \overline{n-1}\,\Delta x)^2\,\Delta x\} \\ &= \underset{\Delta x \doteq 0}{L}\ \Delta x\{1 + (1 + \Delta x)^2 + (1 + 2\,\Delta x)^2 + \dots \\ &\qquad + (1 + \overline{n-1}\,\Delta x)^2\} \\ &= \underset{\Delta x \doteq 0}{L}\ \Delta x\{n + 2\,\Delta x\,(1 + 2 + 3 + \dots + \overline{n-1}) \\ &\qquad + (\Delta x)^2\,(1^2 + 2^2 + 3^2 + \dots + \overline{n-1}^2)\} \quad (5) \end{aligned}$$

From algebra,* we know that

$$1 + 2 + 3 + \dots + (n-1) = \frac{n\,(n-1)}{2},$$

$$1^2 + 2^2 + 3^2 + \dots + (n-1)^2 = \frac{n\,(n-1)\,(2n-1)}{6}.$$

Substituting these values, we have after taking out the factor n,

$$\text{area } A_0P_0QB = \underset{\Delta x \doteq 0}{L}\ n\,\Delta x\left\{1 + n\,\Delta x - \Delta x + (\Delta x)^2\,\frac{(n-1)\,(2n-1)}{6}\right\} \quad (6)$$

$$= \underset{\Delta x \doteq 0}{L}\ n\,\Delta x\left\{1 + n\,\Delta x - \Delta x + \frac{2\,(n\,\Delta x)^2 - 3\,(n\,\Delta x)\,\Delta x + (\Delta x)^2}{6}\right\}. \quad (7)$$

However, we have $n\,\Delta x = AB = 2$; consequently

$$\text{area } A_0P_0QB = \underset{\Delta x \doteq 0}{L}\ 2\left\{1 + 2 - \Delta x + \frac{8 - 6\,\Delta x + (\Delta x)^2}{6}\right\} = 8\tfrac{2}{3}. \quad (8)$$

Comparing this result with the value of the definite integral

$$\int_1^3 y\,dx = \int_1^3 x^2\,dx = \frac{x^3}{3}\Big]_1^3 = 8\tfrac{2}{3},$$

we see that the evaluation of the integral gives in this case the same numerical result as the summation. In the following article, we shall show that in general the summation process can be replaced by the shorter method of the definite integral.

Ex. Find the area bounded by the straight line $y = 3x$, the X-axis, and the ordinates $x = 2, x = 5$. Show that the definite integral $\int_2^5 3x\,dx$ gives the same result.

80. The definite integral as the limit of a sum. Let $f(x)$ be a single-valued continuous function in the interval $a \leqq x \leqq b$, and let the interval (a, b) be divided into n parts by inserting between a and b, in any manner whatever, $n - 1$ values x_1, x_2, . . . x_{n-1} in ascending order of magnitude. Suppose that each subinterval is multiplied by the value of $f(x)$ at some point within it, and that the products thus obtained are added. Denoting the values of $f(x)$ taken within the several subintervals by $f_1(x)$, $f_2(x)$, $f_3(x)$ . . . $f_n(x)$, respectively, the result is

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* See Hall and Knight's *Higher Algebra*, Arts. 67, 68.

$$(x_1-a)\,f_1(x)+(x_2-x_1)\,f_2(x)+\ldots+(b-x_{n-1})\,f_n(x)\,. \tag{1}$$

If we take Δx as a general symbol denoting a subinterval, (1) may be written in the more compact form,

$$\sum_a^b f(x)\,\Delta x.$$

We shall now show that as we make the subintervals smaller and smaller by introducing new points of division within the original subintervals, this sum approaches a limit, and that this limit is the definite integral

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

In any subinterval, as $x_{i+1} - x_i$, let α_i denote the minimum value and β_i the maximum value of $f(x)$. If we multiply each interval by its α and add the products, we obtain the sum

$$S' = \alpha_1(x_1-a) + \alpha_2(x_2 - x_1) + \ldots + \alpha_n(b - x_{n-1})\,. \tag{2}$$

By multiplying each interval by its β and adding, we obtain

$$S'' = \beta_1(x_1 - a) + \beta_2(x_2 - x_1) + \ldots + \beta_n\,(b - x_{n-1}). \tag{3}$$

From the manner of deriving the sums (1), (2), (3), it follows that

$$S' \leqq \sum f(x)\ \Delta x \leqq S''. \tag{4}$$

Subtracting (2) from (3), we obtain

$$S'' - S' = \sum_a^b (\beta_i - \alpha_i)\ (x_{i+1} - x_i). \tag{5}$$

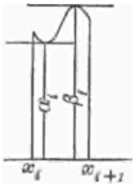

Fig. 59.

The difference $(\beta_i - \alpha_i)$ measures the amount by which $f(x)$ can vary within the interval $x_{i+1} - x_i$. (See Fig. 59.) If E is the largest variation that $f(x)$ can have in any of the subintervals, then from (5)

$$S'' - S' \leqq \sum_a^b E(x_{i+1} - x_i) = E(b - a). \tag{6}$$

We have so far dealt with an arbitrary division of the interval (a, b) into n parts. Let us now consider the effect of increasing the number of parts by introducing in any manner whatever new points of division in the original subintervals. By continuing the process of subdivision, we can make the subintervals as small as we please, that is, we can make Δx approach zero. Now since $f(x)$ is continuous, the amount of its variation in an interval decreases with the size of the interval; hence $E \doteq 0$, as $\Delta x \doteq 0$. From (6) we have therefore

$$\underset{\Delta x \doteq 0}{L}(S'' - S') \leqq \underset{E \doteq 0}{L} E(b - a) = 0;$$

and since $S'' - S'$ is necessarily positive for $\Delta x \neq 0$, it follows that

$$\underset{\Delta x \doteq 0}{L}(S'' - S') = 0. \tag{7}$$

The sums S'' and S' are functions of the kind described in Art. 12, that is, they are monotone functions. Both are finite, and as n increases, S'' never increases and S' never decreases. Hence each sum has a limit, and from (7) the limits are equal. From (4) it follows that $\sum\limits_a^b f(x)\ \Delta x$ has the same limit. (See Art. 11, Th. VI.) We have, therefore,

$$\underset{\Delta x \doteq 0}{L} S'' = \underset{\Delta x \doteq 0}{L} S' = \underset{\Delta x \doteq 0}{L} \sum_a^b f(x)\ \Delta x. \tag{8}$$

The considerations of Art. 82 make it geometrically evident that the same limiting value of $\sum\limits_a^b f(x)\ \Delta x$ is obtained as Δx is made to approach zero by all methods of subdivision of the interval (a, b); that is, the value of the limit is independent of the *manner* of subdivision.*

Having shown that $\sum\limits_a^b f(x)\ \Delta x$ has a limit, we have now to show that this limit is $\int_a^b f(x)\ dx$. In this discussion $f_i(x)$ was taken as *any* value of $f(x)$ within the interval $x_{i+1} - x_i$. Furthermore Δx was made to approach zero by the arbitrary insertion of new points of subdivision in each subinterval. Since the existence of the limit in (8) has been established, we may find its value by taking the subintervals and the corresponding values of the function in any particular manner we please. It is convenient to make the values of Δx equal and to take the value of $f(x)$ at the beginning of each interval. We have then for all values of n,

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* For an analytic proof of this statement, see Picard's *Traité d'Analyse*, Vol. I, p. 4, or Veblen and Lennes' *Infinitesimal Analysis*, p. 154, et seq.

$$n\,\Delta x = b - a, \tag{9}$$

and the original points of division become

$$a + \Delta x, a + 2\,\Delta x, \; . \; . \; . \;, b - \Delta x$$

In defining the definite integral, $\phi(x)$ was taken as a function having $f(x)$ as its derivative. From the law of the mean, we have therefore

$$\phi(x + \Delta x) - \phi(x) = \Delta x \cdot f(x + \theta \cdot \Delta x), 0 < \theta < 1. \tag{10}$$

By hypothesis α and β are respectively the minimum and maximum values of $f(x)$ in a subinterval; hence we may write

$$\alpha \leqq f(x + \theta \cdot \Delta x) \leqq \beta.$$

By aid of (10) this inequality becomes

$$\alpha \cdot \Delta x \leqq \phi(x + \Delta x) - \phi(x) \leqq \beta \cdot \Delta x. \tag{11}$$

We may now apply (11) to the successive subintervals by substituting for x the values

$$a, a + \Delta x, a + 2\,\Delta x, \; . \; . \; . \; . \;, b - \Delta x.$$

The resulting inequalities are

$$\begin{aligned}
&\alpha_1\,\Delta x \leqq \phi(a + \Delta x) - \phi(a) \leqq \beta_1\,\Delta x,\\
&\alpha_2\,\Delta x \leqq \phi(a + 2\,\Delta x) - \phi(a + \Delta x) \leqq \beta_2\,\Delta x,\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\
&\alpha_n\,\Delta x \leqq \phi(b) - \phi(b - \Delta x) \leqq \beta_n\,\Delta x.
\end{aligned}$$

Adding these results, we have

$$\sum_a^b \alpha_i\,\Delta x \leqq \phi\,(b) - \phi(a) \leqq \sum_a^b \beta_i\,\Delta x. \tag{12}$$

If we now make Δx approach zero by dividing the subintervals into equal parts and continuing this process of division, we have

$$\underset{\Delta x \doteq 0}{L} \sum_a^b \alpha_i \Delta x = \underset{\Delta x \doteq 0}{L} S', \text{ and } \underset{\Delta x \doteq 0}{L} \sum_a^b \beta_i \Delta x = \underset{\Delta x \doteq 0}{L} S''.$$

Hence from (12) and (8) we have

$$\phi(b) - \phi(a) = \underset{\Delta x \doteq 0}{L} S' = \underset{\Delta x \doteq 0}{L} S''.$$

We may therefore write

$$\underset{\Delta x \doteq 0}{L} \sum_a^b f(x) \Delta x = \phi(b) - \phi(a) = \int_a^b f(x)\, dx. \tag{13}$$

Because of the identical relation between the limit and the definite integral given in (13), this limit is often made the fundamental definition of such an integral. As has been stated, this limit and hence the definite integral exists if $f(x)$ is continuous. Consequently, whenever $f(x)$ is continuous, we may pass at once from the summation form of expression to that of the definite integral.

81. Value of the summation process. The idea of the summation of an infinite number of infinitely small terms is of fundamental importance in the applications of the calculus to geometry and physics. In the following chapters will be given examples of the use of the summation process in finding the lengths of curves, the areas of surfaces, the volumes of solids, etc. In mechanics, determinations of centers of gravity and moments of inertia also involve the summation principle. The work done by a variable force is found by the summation of terms of the type $F \Delta s$, where F denotes the force and s the distance of the body to which it is applied from some chosen origin. The impulse of a variable force is the summation of terms of the type $F \Delta t$. The space over which a moving point travels is found by the summation of terms of the type $v \Delta t$, where v denotes the velocity of the point. If water flows through a pipe of cross section A with the velocity v, the volume flowing in the time t is Avt provided v is constant during this time interval. If v is not constant, the quantity discharged is found by the summation of terms of the type $Av \Delta t$. If the specific heat c of a substance varies with the temperature τ, the heat that must be imparted to the substance to produce a given rise of temperature is determined by the summation $\Sigma c\Delta\tau$. Other applications will occur to the thoughtful student.

It should be noted that the summation of an infinite number of terms is always necessary when one of the factors entering into the problem varies continuously. As an illustration take the problem of finding the mass of a body. If V denotes the volume of the body and γ its density, the product γV gives the mass, provided the density is constant throughout. The density, however, may differ for different parts of the body, as, for example, when the body is composed of different liquids which arrange themselves in layers or strata. If $V_1, V_2, V_3, \ldots, V_n$ denote the volumes of the separate parts, and $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n$ the corresponding densities, the total mass is evidently the sum

$$\gamma_1 V_1 + \gamma_2 V_2 + \gamma_3 V_3 + \ldots + \gamma_n V_n.$$

In this case, the number of parts being finite, we need only simple addition. However, the density may vary continuously throughout the body, as in the case of the atmosphere. Here we must have recourse to the summation of an infinite number of indefinitely small terms. We divide the total volume V into n parts each equal to ΔV and multiply each element ΔV by the density at that part of the body. We thus get n terms of the type $\gamma\Delta V$. If n is finite, the sum of these n terms is not the exact value of the mass because the density varies in the element of volume ΔV. But as n is taken larger and ΔV smaller, the sum of the n terms approaches more nearly the mass. Hence, to get the exact result, we must increase n indefinitely, thus making ΔV correspondingly small, and effect the summation of the infinitely large number of infinitesimal terms. That is, we must find

$$\underset{\Delta V \doteq 0}{L} \Sigma \gamma \Delta V.$$

Equation (13), Art. 80, gives a convenient method of effecting a summation of an infinite number of infinitesimal elements, the terms being of the type $f(x)\Delta x$. We have only to find the antiderivative $\phi(x)$ of the function $f(x)$, substitute the limits of integration, say a and b, and take the difference $\phi(b) - \phi(a)$. The problem of performing such a summation reduces, therefore, to a problem in integration.

82. Geometric interpretation of a definite integral. When we represent graphically the processes employed in Art. 80, we are led to a useful geometric interpretation of the definite integral

$$\int_a^b f(x)\, dx.$$

Let the curve EF, Fig. 60, be the graph of the single-valued continuous function $f(x)$, and let the interval (a, b) be divided

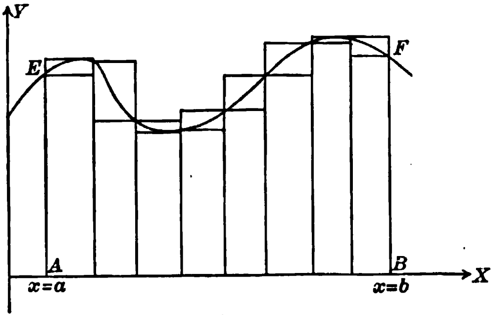

Fig. 60.

into subintervals by locating points between A and B on the X-axis. At these points of division let ordinates be erected. In any subinterval Δx_i let α_i and β_i denote respectively the smallest and largest numerical values of the ordinates of the curve EF. With $\Delta x_i = x_{i+1} - x_i$ as a base, let two rectangles be constructed having α_i and β_i respectively as altitudes. Repeating this construction for each subinterval, we obtain finally two plane areas, the first A' made up of rectangles lying entirely below the curve EF, the other A'' made up of rectangles whose upper bases lie above the curve. We have therefore

$$A' \leqq \text{area } AEFB \leqq A''.$$

The area A' represents graphically the sum $S' = \sum_a^b \alpha_i \Delta x_i$ given by (2), Art. 80; and likewise A'' represents the sum $S'' = \sum_a^b \beta_i \, \Delta x_i$ given by (3). It follows that as Δx is made to approach zero, A' and A'' have the same limit; and from Art. 11, Th. VI,

$$\text{area } AEFB = \underset{\Delta x \doteq 0}{L} A' = \underset{\Delta x \doteq 0}{L} A''.$$

From Art. 80, (12) and (13),

$$\int_a^b f(x)\,dx = \underset{\Delta x \doteq 0}{L}\, S' = \underset{\Delta x \doteq 0}{L}\, S''.$$

Hence, since the area A' represents graphically the sum S', the area $AEFB$, which is the limit of area A', represents graphically the limit of S', that is, $\int_a^b f(x)\,dx$. This result may be formally stated as follows:

The definite integral $\int_a^b f(x)dx$ *is represented by the area between the curve* $y = f(x)$, *the X-axis, and the two ordinates* $x = a$, $x = b$.

In applying this theorem, certain precautions must be observed. For example, if the graph of $f(x)$ crosses the X-axis, as shown in Fig. 61, the integral $\int_a^b f(x)\,dx$ is represented by the algebraic sum of the areas I, II, III, where II appears as a negative area since it lies entirely below the X-axis. If we wish the numerical rather than the algebraic sum of these three areas, we must take the numerical sum of the three integrals

$$\int_a^b f(x)\,dx, \qquad \int_b^c f(x)\,dx, \qquad \int_c^d f(x)\,dx.$$

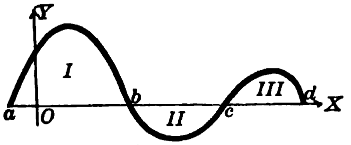

Fig. 61.

We have said that the definite integral *is represented by* a given area. It is not to be inferred that a definite integral *is* such an area. What is meant is that the definite integral $\int_a^b f(x)\,dx$ has the same numerical value as the area in question.

A definite integral may be used to denote the magnitude of such measurable quantities as distance, volume, work, heat, impulse, or mass. But whatever its concrete association in the problem under discussion, that same numerical value is represented by the area mentioned above. Hence, we may, and frequently shall, represent such quantities as those mentioned by their corresponding areas.

EXERCISES.

1. Give a geometric interpretation of Theorem II, Art. 77.

2. Give a geometric proof of the following theorem: If M and N are respectively the greatest and least values of the continuous function $f(x)$ within the interval (a, b), then, provided $b > a$,

$$N(b-a) < \int_a^b f(x)\,dx < M(b-a).$$

3. Give a geometric proof of the following theorem: If $b > a$, and $\phi(x)$, $f(x)$, and $\psi(x)$ are three functions such that for any value of x within the interval (a, b), $\phi(x) < f(x) < \psi(x)$, then

$$\int_a^b \phi(x)\,dx < \int_a^b f(x)\,dx < \int_a^b \psi(x)\,dx.$$

4. Using the theorem of Ex. 3, show that if $0 < n < 2$, $\int_0^1 \frac{dx}{1+x^n}$ lies between 0.5 and .7854, and that if $n > 2$, it lies between 0.7854 and 1.

5. Give a geometric proof of the theorem

$$\int_0^a F(x)\,dx = \int_0^a F(a-x)\,dx.$$

6. Show that
$$\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta.$$

7. Show from geometric considerations that

$$\int_0^{2\pi} \sin\theta\,d\theta = 0.$$

8. Show the area that represents the definite integral $\int_{t_1}^{t_2} gt\,dt$, which applies to falling bodies.

83. Definite integrals of discontinuous functions: Infinite limits of integration. So far we have discussed definite integrals of continuous functions with finite limits of integration. We must now consider cases where one or both of these conditions do not hold.

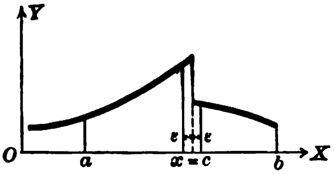

Fig. 62.

Suppose that $f(x)$ has a point of discontinuity at $x = c$, Fig. 62. The extent of this discontinuity may be finite or infinite. The integrals $\int_a^b f(x)\,dx$, $\int_a^c f(x)\,dx$, $\int_c^b f(x)\,dx$ have here no meaning by the definition given for a definite integral. However, the definite integral does have a meaning for the intervals $(a, c-\varepsilon)$,

$(c + \epsilon, b)$, for within these intervals the function is continuous. We may, therefore, define the value of the definite integral for the intervals (a, c) and (c, b) as the limits

$$\underset{\epsilon \doteq 0}{L}\int_a^{c-\epsilon} f(x)\,dx, \qquad \underset{\epsilon \doteq 0}{L}\int_{c+\epsilon}^{b} f(x)\,dx,$$

provided these limits exist. The definite integral for the interval (a, b) may therefore be defined as the sum of these two limits. We may not, however, obtain the proper value of the last integral by evaluating directly the integral $\int_a^b f(x)\,dx$.

Ex. 1. Find the area between the curve $y = \frac{1}{(x-2)^2}$, the X-axis, and the ordinates for which $x = 0$, $x = 4$.

This function is discontinuous for $x = 2$, as is shown in the figure. The definite integral for the given interval is found by taking the sum of the limits

$$\underset{\epsilon \doteq 0}{L}\int_0^{2-\epsilon} \frac{dx}{(x-2)^2} + \underset{\epsilon \doteq 0}{L}\int_{2+\epsilon}^{4} \frac{dx}{(x-2)^2}$$

$$= \underset{\epsilon \doteq 0}{L}\left.\frac{-1}{x-2}\right]_0^{2-\epsilon} + \underset{\epsilon \doteq 0}{L}\left.\frac{-1}{x-2}\right]_{2+\epsilon}^{4}$$

$$= \underset{\epsilon \doteq 0}{L}\frac{1}{\epsilon} - \frac{1}{2} - \frac{1}{2} + \underset{\epsilon \doteq 0}{L}\frac{1}{\epsilon}$$

$$= \underset{\epsilon \doteq 0}{L}\frac{2}{\epsilon} - 1 = \infty.$$

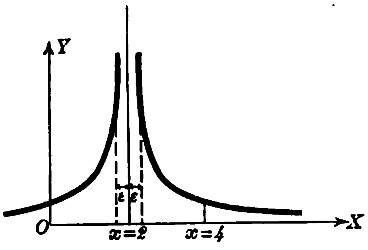

Fig. 63.

Hence the area in question is infinite.

Let us now take the integral directly between the limits 0 and 4. We obtain for the area

$$\int_0^4 f(x)\,dx = \int_0^4 \frac{dx}{(x-2)^2} = \left.\frac{-1}{x-2}\right]_0^4 = -1,$$

an incorrect result. Consequently in this case we say that the integral $\int_0^4 f(x)\,dx$ does not exist.

We must also consider the special case in which one of the limits of integration is infinite. This again is a limiting case of the ordinary definite integral. We define the definite integral

$$\int_a^\infty f(x)\,dx \text{ as the limit } \underset{x = \infty}{L}\int_a^x f(x)\,dx.$$

Hence, provided this limit exists, the integral $\int_a^\infty f(x)\,dx$ has a meaning. The following example will illustrate this case:

Ex. 2. Find the area between the X-axis and the witch of Agnesi, whose equation is

$$y = \frac{8a^3}{x^2 + 4a^2}.$$

See Fig. 49, Art. 71. The curve is symmetrical with respect to the Y-axis and approaches the X-axis as x becomes infinite. Hence the area is given by

$$2\int_0^\infty y\,dx = \underset{x=\infty}{L}\ 16a^3 \int_0^x \frac{dx}{x^2+4a^2} = 16a^3 \underset{x=\infty}{L} \left[\frac{1}{2a}\text{arc tan}\,\frac{x}{2a}\right]_0^x$$

$$= 8a^2 \underset{x=\infty}{L}\ \text{arc tan}\,\frac{x}{2a} = 4\pi a^2.$$

EXERCISES.

1. Evaluate $\int_0^\infty \frac{dx}{a^2+x^2}$.

2. Evaluate $\int_0^\infty e^{-ax}\,dx$.

3. Evaluate $\int_0^1 \frac{dx}{(1-x)^{\frac{1}{2}}}$.

4. Evaluate $\int_{-\infty}^{-1} \frac{dx}{x\sqrt{x^2-1}}$.

5. Of the following definite integrals, which have finite values?

$$\text{(a)}\ \int_0^1 \frac{dx}{x^2};\quad \text{(b)}\ \int_0^1 \frac{dx}{x};\quad \text{(c)}\ \int_a^\infty \frac{dx}{x^3};\quad \text{(d)}\ \int_0^1 \frac{dx}{1-x^2}.$$

6. Show that if $n < 1$, the area under the curve $x^n y = C$ from the origin to $x = a$ is finite, while for $n > 1$ the area under the curve from $x = a$ to $x = \infty$ is finite.

7. With the data of Ex. 6 investigate the area when $n = 1$.

8. Find the area under the curve

$$y\,(x^2-1)^{\frac{2}{3}} = x$$

between $x = 0$ and $x = 4$. Draw the curve.

MISCELLANEOUS EXERCISES.

1. Evaluate the following definite integrals:

$$\text{(a)}\ \int_0^{s_1} \sqrt{2gs}\ ds;\quad \text{(b)}\ \int_0^a \left(x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2}\right)dx;$$

$$\text{(c)}\ \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^3\theta\,d\theta.\quad (\text{Put } \cos^2\theta = 1 - \sin^2\theta.)$$

2. Show that

$$\frac{\int_0^a x(a-x)\,dx}{\int_0^a (a-x)\,dx} = \frac{1}{3}a.$$

3. Evaluate (a) $\dfrac{\int_\alpha^\beta \sin\theta\cos\theta\,d\theta}{\int_\alpha^\beta \sin\theta\,d\theta}$; (b) $\dfrac{\int_0^{\frac{\pi}{2}} a\cos^2\theta\sin\theta\,d\theta}{\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\,d\theta}$

4. Find by the summation method of Art. 79 the area between the curve $y = e^x$, the X-axis, and the ordinates $x = 1$, $x = 4$.

5. A vertical wall (as a dam) is subjected to water pressure, the intensity of which varies as the depth below the liquid surface. By the process of direct summation find the total pressure on the wall.

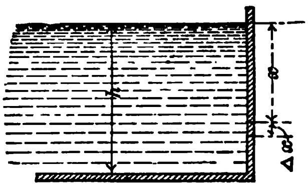

Fig. 64.

Suggestion: The intensity of pressure at a depth x below the surface is kx, where k is constant. Divide the wall into strips of width Δx and length b; then multiply the area of each strip by the pressure at the top of the strip. The terms to be added are then $b\Delta x \cdot 0,\ b\,\Delta x \cdot k\,\Delta x,\ b\,\Delta x \cdot 2\,k\,\Delta x, \ldots, b\,\Delta x \cdot (n-1)\,k\,\Delta x$, where $n\,\Delta x = h$.

6. Change the limits of integration of the following integrals when the variable is changed as indicated.

(a) $\int_4^{20} (\ )\,dx,\quad x = z^2 + 4$; (b) $\int_0^1 (\ )\,dx,\quad x = \sin\theta - \cos\theta$.

(c) $\int_0^{\log 10} (\ )\,dx,\quad e^x = z^2 + 1$; (d) $\int_0^a (\ )\,dx,\quad x = a\sin\theta$.

7. Show from geometrical considerations that

(a) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$; (b) $\int_a^b f(x)\,dx = \int_0^{b-a} f(x+a)\,dx$.

8. Evaluate the following:

(a) $\int_0^2 \dfrac{dx}{\sqrt{2-x}}$; (b) $\int_0^a \dfrac{dx}{\sqrt{a^2-x^2}}$; (c) $\int_0^4 \dfrac{dx}{(x-2)^2}$.

9. Prove and interpret geometrically the following theorems:

(a) If $f(-x) = -f(x)$, $\int_{-a}^{a} f(x)\,dx = 0$.

(b) If $f(-x) = f(x)$, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

10. Apply the theorems of Ex. 9 to the definite integrals

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\, d\theta;\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta.$$

11. If $n > 2$, show that $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^n}}$ lies between 0.5 and 0.52.

12. If $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x + \frac{20}{3}$, show that $\int_0^5 f(x)\, dx$ must lie between 20 and 42.5.

Suggestion: Find maximum and minimum values of $f(x)$ between 0 and 5, and apply the result of Ex. 2, Art. 82.

13. Find values between which $\int_0^{x_1} f(x)\, dx$ must lie, where x_1 is the abscissa of the point of inflexion in the curve of Ex. 12.

14. Given $f''(x) = \frac{d^2y}{dx^2} = 2x - 3$. (a) If the slope is -2 for $x = 0$, find the difference in the ordinates corresponding to $x = 2$, $x = 5$. (b) Find maximum and minimum points of the curve. (c) Find the angle between the tangents to the curve at the points $x = 1$, $x = 4$.

15. Find the radii of curvature of the curves of Ex. 12 and 14 at the point $x = 3$.

16. Prove the following theorem geometrically:

$$\int_a^b \phi(x)\, dx = (b - a)\,\phi[a + \theta\,(b - a)],$$

where $0 < \theta < 1$.

CHAPTER X

APPLICATIONS OF INTEGRATION TO GEOMETRY AND MECHANICS

84. Areas of plane curves, rectangular coördinates. If the equation of a curve in rectangular coördinates is

$$y = f(x), \tag{1}$$

then, as proved in Art. 82, the area between the curve, the X-axis, and the ordinates $x = a$ and $x = b$, respectively, is given by the formula

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx. \tag{2}$$

Ex. 1. Find the area included between the parabola $y^2 = 8x$, the X-axis, the origin, and the ordinate $x = 18$.

Since $y^2 = 8x$, $y = \sqrt{8x}$, and

$$A = \int_0^{18} y\,dx = \sqrt{8}\int_0^{18} x^{\frac{1}{2}}\,dx = \sqrt{8}\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^{18} = 144.$$

Ex. 2. Find the area under one arch of the curve $y = \cos x$.

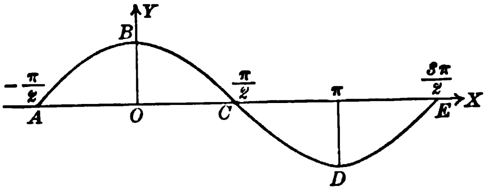

Fig. 65.

From the graph of the curve, Fig. 65, it is seen that $y = 0$ for $x = -\frac{\pi}{2}$ and for $x = \frac{\pi}{2}$. Hence

$$\text{area } ABC = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = \sin x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2.$$

Likewise $$\text{area } CDE = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx = \sin x\,\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -2.$$

Also, $$\text{area } ABCDE = \sin x\,\Big]_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} = 0$$

$$\text{area } BCD = \sin x\,\Big]_0^{\pi} = 0.$$

The last two results are consistent when the signs of the areas are taken into account. See Art. 82. Thus areas ABC and CDE are equal in magnitude but opposite in sign. The absolute magnitude of the area $ABCDE$ is 4.

It will be observed that the area ABC is symmetrical with respect to the axis OY; consequently

$$\text{area } ABC = 2 \times \text{area } OBC = 2\int_0^{\frac{\pi}{2}} \cos x\,dx = 2.$$

Likewise area $$CDE = 2\int_{\frac{\pi}{2}}^{\pi} \cos x\,dx = -2.$$

In general, it is advisable to make use of conditions of symmetry as far as possible, and take the narrowest interval of integration that the problem permits.

Frequently it is desirable to find the area between the curve, the Y-axis, and the abscissas corresponding to $y = \alpha$ and $y = \beta$, respectively. The formula for this area is

$$A = \int_{\alpha}^{\beta} x\,dy. \tag{3}$$

The derivation of form (3) follows precisely that of form (2) and need not be given in detail.

Ex. 3. Find the area between the curve $y^3 = kx$, the Y-axis, and the abscissas for $y = 2$ and $y = 3$.

$$A = \int_2^3 x\,dy = \frac{1}{k}\int_2^3 y^3\,dy = \frac{1}{k}\cdot\tfrac{1}{4}\,y^4\Big]_2^3 = \frac{65}{4k}.$$

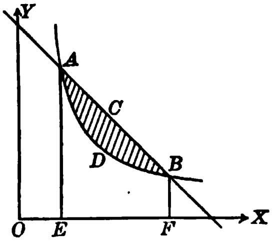

Fig. 66.

Ex. 4. Find the area included between the hyperbola $xy = 36$ and the straight line $x + y = 15$ (Fig. 66). The points of intersection of the two curves are found by solving the equations simultaneously. The solution gives (3, 12) and (12, 3) as the coördinates of A and B, respectively.

$$\begin{aligned}\text{Area } ABCD &= \text{area } EADBF - \text{area } EACBF\\ &= \int_3^{12}(15-x)\,dx - 36\int_3^{12}\frac{dx}{x}\\ &= \left[15x - \frac{x^2}{2} - 36\log x\right]_3^{12} = 17.59.\end{aligned}$$

85. Areas of plane curves, polar coördinates. When polar coördinates are used, the area swept over by the radius vector in passing from an initial position to a final position is the required area.

Let AB, Fig. 67, represent the curve $\rho = f(\theta)$, where $f(\theta)$ is a continuous function. Let OX be the initial line, and α and β the initial and final values of θ. The area required is that of the sector AOB.

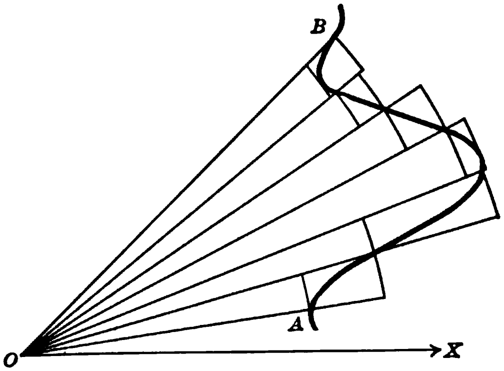

Fig. 67.

Let the angle $AOB = \beta - \alpha$ be divided into n parts, $\Delta\theta_1$, $\Delta\theta_2$, etc. In the ith division, let ρ_i' and ρ_i'' be respectively the smallest and largest values of ρ. If from O as a center arcs are drawn with ρ_i' and ρ_i'' as radii, two circular sectors are formed whose areas are respectively $\frac{1}{2}\rho_i'^2\Delta\theta_i$ and $\frac{1}{2}\rho_i''^2\Delta\theta_i$. Proceeding in this way with each of the n divisions we get n circular sectors with arcs lying within the given curve AB and n circular sectors with arcs lying without the curve. The required area OAB is greater than the sum of the inner sectors and less than the sum of the outer sectors; that is,

$$\sum_{\alpha}^{\beta} \tfrac{1}{2} \rho'^2 \Delta\theta \leqq \text{area } AOB \leqq \sum_{\alpha}^{\beta} \tfrac{1}{2} \rho''^2 \Delta\theta. \tag{1}$$

Since $\sum\limits_{\alpha}^{\beta} \frac{1}{2} \rho'^2 \Delta\theta$ and $\sum\limits_{\alpha}^{\beta} \frac{1}{2} \rho''^2 \Delta\theta$ always remain finite, the first never decreasing and the second never increasing, each sum has a limit, and by the method used in Art. 80, it may be shown that these limits are equal. We have therefore

$$\text{area } AOB = \underset{\Delta\theta \doteq 0}{L} \sum_{\alpha}^{\beta} \tfrac{1}{2} \rho'^2 \Delta\theta = \underset{\Delta\theta \doteq 0}{L} \sum_{\alpha}^{\beta} \tfrac{1}{2} \rho''^2 \Delta\theta. \tag{2}$$

Furthermore, since ρ is a single-valued and continuous function of θ, we may replace the common limit by a definite integral and write

$$A = \tfrac{1}{2} \int_{\alpha}^{\beta} \rho^2 d\theta, \tag{3}$$

where A denotes the required area.

Ex. **1.** Find the area swept over by the radius vector of the curve $\rho = \dfrac{a}{\theta}$, as θ varies from π to 2π.

$$A = \frac{1}{2} \int_0^{2\pi} \rho^2 d\theta = \frac{a^2}{2} \int_{\pi}^{2\pi} \frac{d\theta}{\theta^2} = \frac{a^2}{4\pi}.$$

Ex. **2.** Find the area swept over by the radius vector of the logarithmic spiral $\rho = e^{k\theta}$ in one revolution.

$$A = \frac{1}{2} \int_0^{2\pi} e^{2k\theta} d\theta = \frac{1}{4k} e^{2k\theta} \Big]_0^{2\pi} = \frac{1}{4k} (e^{4k\pi} - 1).$$

EXERCISES.

1. Find the area between the line $y = 3x + 4$, the X-axis, and the lines $x = 0$, $x = 6$.

2. Find the area between the parabola $y = 5x^2$, the X-axis, and the lines $x = 0$, $x = 4$.

3. Find the area under the curve $5y^2 = x^3$ from the origin to the line $x = 5$.

4. Find the area between the parabola $y = 3x^2$ and the straight line $y = 12x$.

5. Take the segment of the equilateral hyperbola $xy = C$ between the points A and B. Show that the area between this segment and the X-axis is the same as the area between the same segment and the Y-axis.

6. Find the area under the catenary $y = \cosh \frac{x}{a}$ between $x = 0$ and $x = m$.

7. Derive a general expression for the area under the curve $xy^m = C$ between $x = a$ and $x = b$. Discuss the case $m = 1$.

8. Find the area swept over by the radius vector of the spiral of Archimedes $\rho = a\theta$ in one revolution.

9. Find the area of the two loops of the lemniscate $\rho^2 = a \cos 2\theta$. Make use of the symmetry of the curve and take narrowest limits of integration.

10. Find the area swept over in two revolutions by the radius vector of the parabolic spiral $\rho^2 = a^2\theta$.

86. Volumes of solids of revolution. A plane area AEFB, Fig. 68, bounded by the curve EF, whose equation is $y = f(x)$, the axis of X and the ordinates AE and BF, rotates about the axis OX and thereby generates a solid of revolution. To derive an expression for the volume of this solid we proceed as follows: Let the interval AB be divided into n parts, Δx_1, Δx_2, Δx_3, etc., and let ordinates be erected at the points of division. In any subinterval Δx_i, let y_i' and y_i'' be respectively the smallest and largest numerical values of the ordinate, and construct rectangles having Δx_i as a base and y_i' and y_i'', respectively, as altitudes. Repeating this construction for each of the subintervals we obtain one plane area made up of rectangles lying entirely below the given curve EF and a second plane area made up of rectangles whose upper bases lie above this curve. The solids obtained by

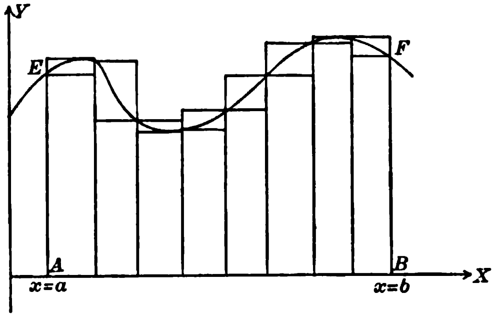

Fig. 68.

revolving these areas about the X-axis have respectively the volumes

$$V' = \sum_a^b \pi y'^2 \Delta x,$$

$$V'' = \sum_a^b \pi y''^2 \Delta x.$$

The volume V of the solid generated by the revolution of the area $AEFB$ must lie between V' and V''. That is,

$$\sum_a^b \pi y'^2 \Delta x \leqq V \leqq \sum_a^b \pi y''^2 \Delta x. \qquad (1)$$

It will be seen that V' and V'' are both monotone functions, the first never decreasing and the second never increasing; hence since the functions are finite, each has a limit as $\Delta x \doteq 0$ (Art. 12). Following the methed of Art. 80, it may be shown that the two limits are equal. Consequently we may write

$$V = \underset{\Delta x \doteq 0}{L} \sum_a^b \pi y'^2 \Delta x = \underset{\Delta x \doteq 0}{L} \sum_a^b \pi y''^2 \Delta x. \qquad (2)$$

Since y is taken to be considered a single-valued and continuous function of x, we may replace the common limit by the definite integral and write

$$V = \pi \int_a^b y^2 \, dx. \qquad (3)$$

By a similar process it can be shown that the volume of the solid generated by a rotation about the Y-axis is

$$V = \pi \int_c^d x^2 \, dy, \qquad (4)$$

where c and d are the ordinates of the end-points of the curve.

Ex. 1. Find the volume generated by the revolution about the X-axis of the area bounded by the line $4x + y = 12$ and the coördinate axes.

For $y = 0$, $x = 3$; hence the limits of integration are $x = 0$, and $x = 3$.

$$V = \pi \int_0^3 y^2 dx = \pi \int_0^3 (12 - 4x)^2 \, dx = \pi \left(144x - 48x^2 + \frac{16}{3}x^3\right)\Big]_0^3 = 144\pi.$$

Ex. 2. Find the volume generated by the rotation about the X-axis of the area bounded by the segment of the parabola $y^2 = 8x$ between the origin, the X-axis, and the ordinate for $x = 6$.

$$V = \pi\int_0^6 y^2\,dx = \pi\int_0^6 8x\,dx = 4\pi x^2\Big]_0^6 = 144\pi.$$

The rotation about the Y-axis of the area bounded by the Y-axis, the corresponding abscissa, and the same segment generates the volume

$$V = \pi\int_0^{\sqrt{48}} x^2\,dy = \pi\int_0^{\sqrt{48}} \frac{y^4}{64}\,dy = \frac{\pi y^5}{320}\Big]_0^{\sqrt{48}} = \frac{\pi}{320}\,(48)^{\frac{5}{2}} = 156.72.$$

EXERCISES.

1. A point (m, n) is joined to the origin by a straight line. Find by integration (a) the volume of the cone generated by revolving the line about the axis OX; (b) about the axis OY.

2. Find the volume of the solid generated by the revolution about the X-axis of a segment of the equilateral hyperbola $xy = 12$ between $x = 4$ and $x = 16$.

3. Find the volume generated by the same segment about the Y-axis.

4. The segment of the curve $y = \sec x$ between $x = 0$ and $x = \frac{1}{4}\pi$ is revolved about the X-axis. Find the volume of the solid generated.

5. Find the volume generated by the revolution of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

about the X-axis, taking $x = 0$ and $x = c$ as limits.

87. Volumes determined by the summation of thin slices. The method of determining a volume by taking the limit of a number of thin slices, which we have used in finding the volume of revolution, can be extended to other solids. We conceive the solid to be cut by a number of parallel planes usually perpendicular to one of the coördinate axes. In this way, the solid is divided into a number of thin slices lying along the axis in question. Suppose the cutting planes to be chosen perpendicular to the X-axis; the thickness of the slice may be denoted by Δx, and the area of the cross section will be a function of x, say $F(x)$. As in Art. 86, the volume V lies between two sums $V' = \sum_a^b \alpha\ \Delta x$ and $V'' = \sum_a^b \beta\ \Delta x$, where α and β denote respectively the smallest and largest values of $F(x)$ within the subinterval Δx; and, furthermore, these sums approach the same limit as $\Delta x \doteq 0$. Hence we have

$$V = \underset{\Delta x \doteq 0}{L} \sum_a^b F(x)\,\Delta x = \int_a^b F(x)\,dx. \quad \text{[Art. 80.]} \tag{1}$$

The limits of integration must be so chosen as to include all the slices.

It will be observed that (1) has the same form as the formula for an area. The $F(x)$ which gives the area of the cross section is, however, generally different from the $f(x)$ which gives the ordinate of the bounding curve.

Ex. 1. A circular cylinder of length h is cut by a plane which passes through the diameter of one base and is tangent to the other base. Required the volume of the piece cut from the cylinder.

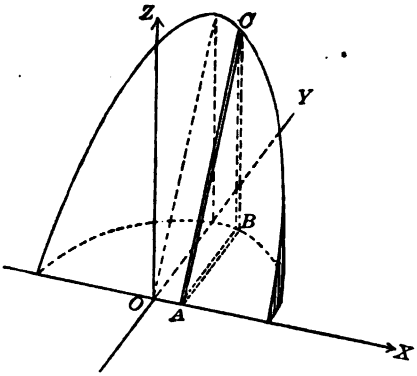

Fig. 69.

Taking the axes as shown in Fig. 69, let the solid be divided into slices by planes parallel to the YZ-plane. The sections cut by the planes are evidently similar triangles. For any triangle as ABC, $AB = y$ and $BC = \frac{y}{a}h$ where a is the radius of the base; hence the area is

$$\frac{y^2 h}{2a} = \frac{(a^2 - x^2)\,h}{2a}.$$

For the total volume we have therefore

$$V = \frac{h}{2a} \cdot 2 \int_0^a (a^2 - x^2)\,dx = \frac{2}{3}a^2 h.$$

Ex. 2. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let the cutting planes be taken parallel to the YZ-plane; then a plane section at a distance x from the YZ-plane will be an ellipse having for its equation

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}.$$

This equation may be written

$$\frac{y^2}{\frac{b^2}{a^2}(a^2 - x^2)} + \frac{z^2}{\frac{c^2}{a^2}(a^2 - x^2)} = 1.$$

The semiaxes of the ellipse are therefore $\frac{b}{a}\sqrt{a^2 - x^2}$ and $\frac{c}{a}\sqrt{a^2 - x^2}$, and the area of the ellipse is $\frac{\pi bc}{a^2}(a^2 - x^2)$.

Hence, we have

$$V = \frac{2\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx = \frac{4}{3}\pi abc.$$

EXERCISES.

1. By the method of integration derive the formula for the volume of a pyramid or cone.

2. Derive the formula for the volume of the frustum of a pyramid or cone.

3. Find the volume of the elliptic paraboloid $\frac{y^2}{4} + \frac{z^2}{9} = 4x$ between the planes $x = 0$ and $x = 3$.

4. Find the volume of the solid bounded by the surface

$$\frac{x^4}{a^4} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

5. A right circular cylinder of base radius a and altitude h has two slices cut from it by planes passing through a diameter of one base and touching the other base. Find the volume of the wedge-shaped solid remaining.

88. Lengths of curves, rectangular coördinates. The length of a curve may be defined as the limit of the sum of the lengths of the inscribed chords as the number of the chords is increased without limit and the length of each approaches the limit zero.

Given the curve whose equation is $y = f(x)$, where $f(x)$ is a continuous function having a continuous derivative. The length s of this curve between the limits $x = a$, $x = b$ is required. Divide the given interval AB, Fig. 70, into n equal parts, denoting by Δx one of these equal divisions. Denote by Δc, Δy, respectively,

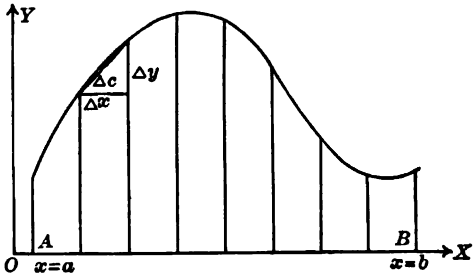

Fig. 70.

the corresponding values of the chord and of the increment of y. From the definition of the length of a curve, we may now write

$$s = \underset{\Delta x \doteq 0}{L} \sum_a^b \Delta c = \underset{\Delta x \doteq 0}{L} \sum_a^b \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x. \qquad (1)$$

With the restrictions imposed upon $y = f(x)$, the quotient $\frac{\Delta y}{\Delta x}$, by the law of the mean, is equal to the derivative $\frac{dy}{dx}$ for some value of x within the subinterval Δx. Hence, since by Art. 80 we may take the value of the integrand $\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$ at any point within this subinterval, we may in (1) replace $\frac{\Delta y}{\Delta x}$ by $\frac{dy}{dx}$ and write

$$s = \underset{\Delta x \doteq 0}{L} \sum_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, \Delta x.$$

Replacing the summation by the definite integral, we have finally

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad (2)$$

If y is taken as the independent variable, this formula for the length of a curve becomes

$$s = \int_\alpha^\beta \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy, \qquad (3)$$

where α, β are the values of y corresponding to $x = a$, $x = b$, respectively.

Ex. 1. Find the length of the curve $y^3 = 3x^2$ from the point (0, 0) to the point (3, 3).

From the given equation we have

$$\frac{dx}{dy} = \frac{y^2}{2x},$$

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{4x^2 + y^4}{4x^2} = \tfrac{1}{4}(4 + 3y),$$

$$s = \int_0^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \tfrac{1}{2}\int_0^3 \sqrt{4 + 3y}\, dy = \tfrac{1}{9}(4 + 3y)^{\frac{3}{2}}\Big]_0^3$$

$$= \tfrac{1}{9}\left(13^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) = 4.32.$$

Ex. 2. Find the length of the cycloid described by a point on the circumference of a rolling disk during one revolution of the disk.

The equations of the cycloid are

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta),$$

in which θ denotes the angle through which the disk has turned.

Differentiating, we have

$$dx = a(1 - \cos\theta)\,d\theta,$$
$$dy = a\sin\theta\,d\theta,$$

whence
$$dx^2 + dy^2 = 2a^2(1 - \cos\theta)\,d\theta^2.$$

From (3), we may write

$$s = \int\sqrt{dx^2 + dy^2},$$

whence
$$s = \int(dx^2 + dy^2)^{\frac{1}{2}} = a\sqrt{2}\int(1 - \cos\theta)^{\frac{1}{2}}\,d\theta.$$

Now $\sqrt{1 - \cos\theta} = \sqrt{2}\sin\frac{1}{2}\theta$, and the limits of θ are obviously 0 and 2π for one revolution.

Hence
$$s = 2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta = -4a\cos\frac{\theta}{2}\Big]_0^{2\pi} = 8a.$$

EXERCISES.

1. Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

2. Find the length of the semicubical parabola $ay^2 = x^3$ from the origin to the point (m, n).

3. Find the length of the catenary from $x = 0$ to $x = x_1$. Use the exponential equation

$$y = \frac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

4. Find the circumference of a circle (a) using the equation $x^2 + y^2 = a^2$; (b) using the equations $x = a\cos\theta$, $y = a\sin\theta$.

5. The involute of a circle is given by the equations

$$x = a(\cos\theta + \theta\sin\theta),$$
$$y = a(\sin\theta - \theta\cos\theta).$$

Find the length of an arc between the limits $\theta = 0$ and $\theta = \pi$.

89. Lengths of curves, polar coördinates. The formulas of the previous article for finding the length of a curve in rectangular coördinates may be easily so transformed as to cover the case of polar coördinates by means of the equations

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta. \tag{1}$$

From Art. 88, we have

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \tag{2}$$

or introducing the differential dx under the radical,

$$s = \int_a^b \sqrt{dx^2 + dy^2}. \tag{3}$$

From equation (1), we obtain

$$\left.\begin{aligned} dx &= -\rho \sin\theta\, d\theta + \cos\theta\, d\rho \\ dy &= \rho\cos\theta\, d\theta + \sin\theta\, d\rho \end{aligned}\right\}. \tag{4}$$

Substituting these expressions in (3), we have

$$\begin{aligned} \sqrt{dx^2 + dy^2} &= \sqrt{[(-\rho\sin\theta\, d\theta + \cos\theta\, d\rho)^2 + (\rho\cos\theta\, d\theta + \sin\theta\, d\rho)^2]} \\ &= \sqrt{\rho^2\, d\theta^2 + d\rho^2}. \end{aligned} \tag{5}$$

Consequently, we have

$$s = \int_\alpha^\beta \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta, \tag{6}$$

where α, β are the limits of integration corresponding to the limits a, b when written in rectangular coördinates.

When ρ is taken as the independent variable, this formula may be written

$$s = \int_{\rho_1}^{\rho_2} \sqrt{1 + \rho^2\left(\frac{d\theta}{d\rho}\right)^2}\, d\rho. \tag{7}$$

Ex. Find the whole length of the cardioid

$$\rho = 2a(1 - \cos\theta).$$

Differentiating the equation of the curve, we get

$$\frac{d\rho}{d\theta} = 2a\sin\theta.$$

Substituting in (6), we have

$$\begin{aligned} s &= 2\int_0^\pi 2a\Big[(1 - \cos\theta)^2 + \sin^2\theta\Big]^{\frac{1}{2}}\, d\theta. \\ &= 2\int_0^\pi 4a\sin\frac{\theta}{2}\, d\theta = 16a\Big[-\cos\frac{\theta}{2}\Big]_0^\pi = 16a. \end{aligned}$$

EXERCISES.

1. Find the length of the circumference of the circle

$$\rho = 2a\cos\theta.$$

2. Find the length of the logarithmic spiral $\rho = e^{a\theta}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$; also from $\theta = \frac{\pi}{2}$ to $\theta = \pi$.

3. Find the length of the spiral $\rho = e^{a\theta}$ from the pole ($\rho = 0$) to $\rho = 1$.

4. Find the length of an arc of the conchoid $\rho = a \sec \theta$ between $\theta = 0$ and $\theta = \frac{\pi}{4}$.

90. Areas of surfaces of revolution. Let a plane curve $y = f(x)$ revolve about the X-axis and thereby generate a surface. We may find an expression for the area S of this surface by a method similar to that employed in finding the length of the curve.

Divide the interval AB, Fig. 71, into n equal parts, and denote the length of each by Δx. Denote by Δc the length of the corresponding chord PQ, by Δy the corresponding increment of y, and by $\Delta S'$ the surface generated by revolving the chord Δc about the axis of X. We have then for the element of surface thus generated

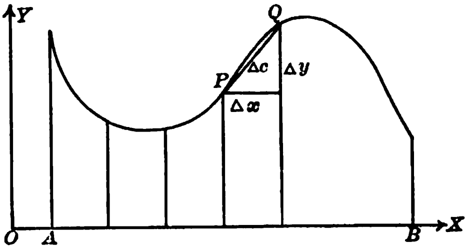

Fig. 71.

$$\Delta S' = 2\pi\left(y \pm \frac{\Delta y}{2}\right)\Delta c. \tag{1}$$

But
$$\Delta c = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x; \tag{2}$$

hence, taking the sum of all the elements of surface thus formed, we have

$$\sum \Delta S' = \sum 2\pi\left(y \pm \frac{\Delta y}{2}\right)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x. \tag{3}$$

As $\Delta x \doteq 0$, the left-hand member of this equation approaches the area of the surface generated by the revolution of the given curve, that is, the required area. In the right-hand member $\left(y \pm \frac{\Delta y}{2}\right)$ approaches y, and $\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$ becomes $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, since $y = f(x)$

is assumed to be a continuous function having a continuous derivative. We have then

$$S = \underset{\Delta x \doteq 0}{L} \sum_a^b 2\pi \left(y \pm \frac{\Delta y}{2}\right) \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x, \tag{4}$$

or

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad \text{(Arts. 80, 88.)} \tag{5}$$

When OY is taken as the axis of revolution, the formula becomes

$$S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \tag{6}$$

From (2) it is evident that we may, if we choose, replace

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

in (5) and (6) by

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

In some problems the latter form is more convenient. The limits of integration must be changed to the values of y corresponding to $x = a,\ x = b$.

Ex. 1. Find the surface generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the X-axis.

From the given equation, we get

$$1 + \left(\frac{dx}{dy}\right)^2 = \left(\frac{a}{y}\right)^{\frac{2}{3}}.$$

We have therefore

$$S = 2 \cdot 2\pi \int_0^a y \left(\frac{a}{y}\right)^{\frac{1}{3}} dy = 4\pi a^{\frac{1}{3}} \cdot \frac{3}{5} y^{\frac{5}{3}}\Big]_0^a = \frac{12}{5}\pi a^2.$$

Ex. 2. Find the area of the surface generated by the revolution of a cycloid about its base.

The equations of the cycloid are

$$x = a\,(\theta - \sin\theta), \quad y = a\,(1 - \cos\theta).$$

Differentiating these equations, we have

$$dx = a\,(1 - \cos\theta)\, d\theta,$$
$$dy = a \sin\theta\, d\theta,$$

whence

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{dx^2 + dy^2} = a\sqrt{2\,(1 - \cos\theta)}\, d\theta.$$

Substituting in (5), we get

$$S = 2\pi a^2 \int_0^{2\pi} \sqrt{2(1-\cos\theta)^3}\, d\theta$$

$$= 16\pi a^2 \int_0^{2\pi} \sin^3\left(\frac{\theta}{2}\right) d\left(\frac{\theta}{2}\right) = \frac{64}{3}\pi a^2.$$

EXERCISES.

1. Find the area of the surface generated by the revolution of the parabola $y^2 = 8x$ about the X-axis. Take the limits $x = 0$ and $x = 2$; also the limits $x = 2$ and $x = 8$.

2. A line joins the origin to the point (m, n). Find by integration the surface of the cone generated by revolving this line about the X-axis.

3. Find the area generated by revolving about the X-axis the arc of the cubical parabola $2y = x^3$ between $x = 0$ and $x = 2$.

4. Find the area of the surface generated by revolving about the X-axis the arc of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ between $x = 0$ and $x = a$.

91. Mean value. Let $y = f(x)$ be a continuous function within the interval $x = a$ and $x = b$, and suppose this interval to be divided into n equal parts each equal to Δx. Then $b - a = n\,\Delta x$. Denoting by $y_1, y_2, y_3, \ldots, y_n$ the values of the function corresponding to the values of x at the middle points of these successive subdivisions, let us form the quotient

$$\frac{y_1 + y_2 + y_3 + \ldots + y_n}{n}, \tag{1}$$

which evidently is merely the arithmetic mean of the n values of y. This quotient will vary with the number of divisions n, and the limit which it approaches as n is indefinitely increased is called the mean value of the function for the interval $b - a$.

Substituting for n its value $\frac{b-a}{\Delta x}$, (1) may be written

$$\frac{y_1\,\Delta x + y_2\,\Delta x + \ldots + y_n\,\Delta x}{b - a}.$$

The limiting value of this quotient as n becomes infinite, and consequently as $\Delta x \doteq 0$, is

$$\frac{\underset{\Delta x \doteq 0}{L} \sum_a^b y\,\Delta x}{b - a} = \frac{\int_a^b y\,dx}{b - a}. \tag{2}$$

The numerator of (2) is represented by the area under the curve $y = f(x)$ (as $AEFB$, Fig. 72) between the ordinates for $x = a$ and $x = b$. Hence if a rectangle $AMNB$ be constructed having an area equal to the area under the curve, the altitude of this rectangle represents the quotient

$$\frac{\int_a^b y\,dx}{b - a}$$

or the mean value of the function.

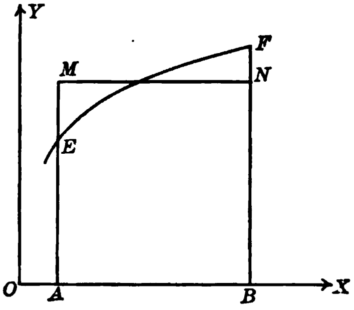

Fig. 72.

The independent variable may be time, distance, angle, area, volume, or any other geometrical or physical magnitude. Mean values may be taken with reference to different variables. Thus in the case of a moving point, values of the velocity may be taken for equal time intervals or for equal space intervals. In the former case, the mean velocity will be the mean ordinate of the velocity curve on a time base, in the latter case it will be the mean ordinate of the velocity curve on a space base.

Ex. 1. In simple harmonic motion the velocity of the moving point is

$$v = a \sin \omega t,$$

and the time of a half-oscillation is $t = \pi/\omega$.

Hence, the mean velocity for the time interval 0 to π/ω is

$$\frac{a\int_0^{\frac{\pi}{\omega}} \sin \omega t\,dt}{\frac{\pi}{\omega} - 0} = \frac{\left[-\frac{a}{\omega}\cos \omega t\right]_0^{\frac{\pi}{\omega}}}{\frac{\pi}{\omega}} = \frac{2}{\pi}a.$$

Ex. 2. In the case of a falling body we have

$$v = gt$$

and

$$v^2 = 2gs.$$

For the mean velocity, taking time as the variable, we have for the interval 0 to t_1,

$$\frac{\int_0^{t_1} v\,dt}{t_1 - 0} = \frac{g\int_0^{t_1} t\,dt}{t_1} = \frac{1}{2}gt_1.$$

Since the velocity at the end of the time t_1 is gt_1, the mean velocity is one-half the final velocity. This fact is also evident from the velocity curve on a time base, Fig. 2.

The mean velocity for the space s_1, taking equal space intervals, is

$$\frac{\int_0^{s_1} v\,ds}{s_1 - 0} = \frac{\sqrt{2\,g}\int_0^{s_1}\sqrt{s}\,ds}{s_1} = \frac{2}{3}\sqrt{2\,gs_1} = \frac{2}{3}v_1;$$

that is, the mean velocity is two-thirds the final velocity.

EXERCISES.

1. Find the mean of the ordinates of the parabola $y^2 = 10\,x$ from $x = 0$ to $x = 8$.

2. Find the mean ordinate of the curve $y = x^2 - 7\,x + 5$ between $x = 1$ and $x = 5$.

3. Find the mean value of the ordinates of the curve $y = \cos x$ (a) between $x = 0$ and $x = \frac{1}{2}\pi$; (b) between $x = 0$ and $x = \pi$.

4. In the compression of air, the pressure rises as the volume of air decreases, as shown by curve AB, then as the air is expelled from the compressor cylinder, the pressure is constant, as shown by BC. Derive an expression for the mean pressure throughout the stroke exerted by the compressor piston on the air. Take $pv = C$ for the curve AB and denote the volumes at A and B by v_a and v_b respectively.

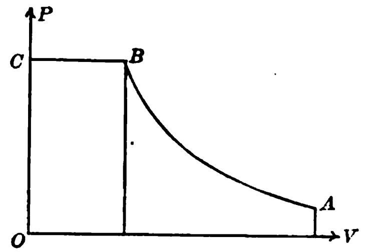

Fig. 73.

5. In Fig. 73, let $v_a = 20$ cu. ft., $v_b = 5$ cu. ft., and $p_a = 14.5$ lb. per sq. in. Find the mean pressure.

92. Work of a variable force. When the point of application of a force is moved in the direction of the line of action of the force, the force is said to do *work*. For example, work is done by the draw-bar pull of a locomotive when the locomotive moves along the track, thus moving the point of application.

If the force is constant in magnitude, the work done is the product of the force and the distance through which the point of application moves in the direction of the force. If W denotes the work, F the force, and s the displacement of the point of application, then

$$W = Fs. \qquad (1)$$

For example, the work done in raising a load of 800 lb. a height of 6 ft. is 800 × 6 = 4800 foot-pounds.

If the direction of the displacement s does not coincide with the line of action of the force, the work done is the product of the force and the component of s in the direction of F. Let θ denote the angle between the directions of the displacement and line of action; then

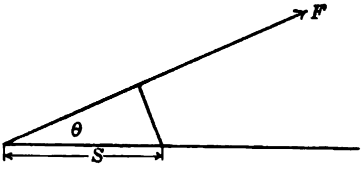

Fig. 74.

$$W = Fs \cos \theta. \tag{2}$$

The magnitude of the force may vary as the point of application moves, as in compressing a spring. In this case, the work may be expressed as a definite integral as follows: Let s_1 and s_2 be the initial and final distances of the point of application of the force from some origin, so that $s_2 - s_1$ is the displacement. Let $s_2 - s_1$ be divided into n subintervals Δs_1, Δs_2, etc., and in any subinterval Δs_i let F_i' and F_i'' be respectively the smallest and largest values of the force F. Then denoting the work by W, we have

$$\sum_{s_1}^{s_2} F' \Delta s \leqq W \leqq \sum_{s_1}^{s_2} F'' \Delta s. \tag{3}$$

As in Art. 80, we may show that the finite monotone functions $\sum_{s_1}^{s_2} F' \Delta s$ and $\sum_{s_1}^{s_2} F'' \Delta s$ have the same limit as $\Delta s \doteq 0$, and we may therefore write at once,

$$W = \underset{\Delta s \doteq 0}{L} \sum_{s_1}^{s_2} F \ \Delta s = \int_{s_1}^{s_2} F \, ds. \tag{4}$$

From the method of deriving (4), it is clear that the work of a force may be represented by an area. On the displacement S_1S_2 let the forces F corresponding to the successive positions of the point of application be laid off as ordinates, and let a curve be drawn through the ends of these ordinates (see Fig. 75), then the area S_1ABS_2 under this curve will represent the work of the force between S_1 and S_2. For the area under the curve is

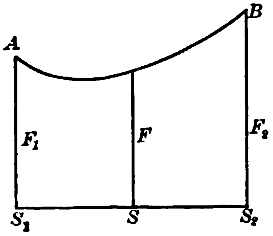

Fig. 75.

$$\int_{s_1}^{s_2} F \, ds,$$

which according to (3) gives the work W.

In order to integrate the expression $F\,ds$, the force F must be expressed as a function of the displacement s. The following cases are those occurring most frequently in practice.

(a) *When the force varies directly as the displacement.*

This is the law in the compression of a spring. We have

$$F = ks + b,$$

$$W = \int_{s_1}^{s_2} (ks + b)\,ds = \frac{1}{2}\,ks^2 + bs\Big]_{s_1}^{s_2}$$

$$= \frac{1}{2}\,k\,(s_2^2 - s_1^2) + b\,(s_2 - s_1).$$

Ex. 1. A spring is 10 inches long, and a force of 48 pounds is required for each inch it is compressed. Find the work of compressing the spring from 10 inches to a length of 6 inches; also from a length of 8 inches to a length of 5 inches.

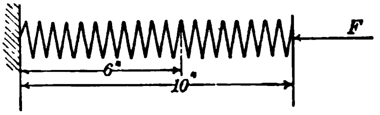

Fig. 76.

In the first case,

$$s_1 = 0,\ s_2 = 10 - 6 = 4,\ F = 48\,s.$$

Hence $$W = \int_{s_1}^{s_2} F\,ds = 48\int_0^4 s\,ds = 24\,s^2\Big]_0^4 = 384 \text{ in. lb.}$$

For the second case, $s_1 = 2$, $s_2 = 5$, and

$$W = 48\int_2^5 s\,ds = 24\,s^2\Big]_2^5 = 504 \text{ in. lb.}$$

Ex. 2. A bar is stretched from its original length L by a gradually increasing load. Denoting by s the amount of stretch for a given force F, Hooke's law gives as the relation between F, s, and L,

$$F = \frac{EAs}{L},$$

where E denotes the coefficient of elasticity of the material, and A the area of the bar. The work of stretching the bar by an amount s is therefore

$$W = \int_0^a F\,ds = \frac{EA}{L}\int_0^a s\,ds = \frac{EAa^2}{2\,L}.$$

As a numerical example, take $E = 30{,}000{,}000$ for wrought iron, and suppose a bar having a cross section of 2 square inches be stretched from 60 inches to 60.5 inches. Here $A = 2$, $a = 0.5$, and $L = 60$; hence the work is

$$W = \frac{30{,}000{,}000 \times 2 \times 0.5^2}{2 \times 60} = 125{,}000 \text{ in. lb.}$$

(b) *When the force varies inversely as the displacement.*

In this case,

$$F = \frac{k}{s},$$

whence $$W = \int_{s_1}^{s_2} F\,ds = k\int_{s_1}^{s_2} \frac{ds}{s} = k \log \frac{s_2}{s_1}.$$

(c) *When the force varies inversely as the square of the distance.*

The law of the inverse square applies to gravitational forces, to forces between electric charges, etc.

Ex. Let a positive charge m of electricity be concentrated at a point P, and a unit charge at a point Q at a distance s from P; then the repulsion of charge m on the unit charge is

$$F = \frac{m}{\overline{PQ}^2} = \frac{m}{s^2}.$$

The work required to move the unit charge from $s = a$ to $s = b$ is

$$W = \int_a^b F\,ds = m\int_a^b \frac{ds}{s^2} = -m\,\frac{1}{s}\Big]_a^b = m\left(\frac{1}{a} - \frac{1}{b}\right).$$

93. Work of expanding gases. A gas is confined by the walls of a cylinder and a movable piston, Fig. 77. By virtue of its pressure, the gas expands or increases in volume, moves the piston, and thus does work against some external resistance. Let p denote the pressure exerted by the gas on a unit area (square inch or square foot), and A the area of the piston. Evidently the total force acting on the piston is $F = pA$; and for $F_1\Delta s$, $F_2\Delta s$, etc., we may write $p_1A\,\Delta s$, $p_2A\,\Delta s$, etc. But $A\,\Delta s$ is the volume swept over by the piston in moving through the distance Δs, and may be denoted by Δv; hence $F\,\Delta s = p\,\Delta v$, and the work done is

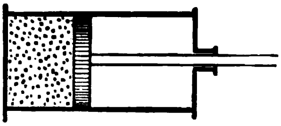

Fig. 77.

$$W = \underset{\Delta v \doteq 0}{L} \sum_{v_1}^{v_2} p\,\Delta v = \int_{v_1}^{v_2} p\,dv. \qquad \text{(Art. 80.)}$$

In using this formula it must be noted (1) that p denotes pressure per unit area, not the total pressure, and (2) that for correct numerical results consistent units must be used, pounds and feet, or pounds and inches throughout; thus, if p is in pounds per square *inch*, v must be in cubic *inches*, and the result will be work in *inch*-pounds.

Ex. **1.** Air expands without change of temperature following Boyle's law. $pv = p_1v_1 = \text{const.}$ The work of expansion is

$$W = \int_{v_1}^{v_2} p\,dv = p_1v_1 \int_{v_1}^{v_2} \frac{dv}{v} = p_1v_1 \log \frac{v_2}{v_1}\,.$$

Ex. **2.** Air expanding *adiabatically* follows the law

$$pv^k = p_1v_1^k = \text{const., where } k = 1.41.$$

The work done during the expansion is

$$W = \int_{v_1}^{v_2} p\,dv = p_1v_1^k \int_{v_1}^{v_2} v^{-k}\,dv = p_1v_1^k \left[\frac{v^{1-k}}{1-k}\right]_{v_1}^{v_2}$$

$$= \frac{p_1v_1^k}{k-1}\left[v_1^{1-k} - v_2^{1-k}\right] = \frac{p_1v_1 - p_2v_2}{k-1},$$

since

$$p_1v_1^k = p_2v_2^k.$$

EXERCISES.

1. The unstretched length of a spring is 16 inches, and a force of 225 pounds is required to stretch it 1 inch. Find the work required to stretch it from a length of 18 inches to a length of 22.5 inches.

2. In hoisting coal or ore from a mine, the load consists of two parts: (1) the weight M of the car and contents; (2) the weight of the rope, which is m pounds per foot. Find the work required for hoisting a distance of h feet.

Suggestion: Let s denote the distance of the load from the lower level; then $F = M + m\,(h - s)$, and

$$W = \int_0^h [M + m\,(h - s)]\,ds.$$

3. In Exs. 1 and 2 draw diagrams showing by areas the work done, and derive the results by elementary geometry.

4. Suppose the force to vary directly as the square of the displacement of its point of application. Derive a formula for the work.

5. Confined air having a volume of 6 cubic feet and a pressure of 80 pounds per square inch expands following the law $pv = \text{const.}$ to a final volume of 20 cubic feet. Find the work done.

6. Use the data of Ex. 5 and find the work done if the expansion is adiabatic, i.e., according to the law $pv^{1.41} = \text{const.}$

7. Find the work of stretching a round iron bar having a diameter of 1.5 inches from a length of 40 inches to a length of 42.3 inches. Take $E = 28{,}000{,}000$.

MISCELLANEOUS EXERCISES.

1. Find the areas bounded by the following curves, the X-axis, and the ordinates indicated:

(a) $y = x^3 + x + 5$, from $x = 0$ to $x = 6$.
(b) $y = x^3 - 3x^2 + 4$, from $x = 1$ to $x = 5$.
(c) $y = e^x$, from $x = 0$ to $x = 1$.

2. Find the area of one loop of the curve $\rho = a \sin 2\theta$.

3. Find the area between the curve $y = \tan x$ and the X-axis from $x = 0$ to $x = \frac{\pi}{3}$.

4. Find the area between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes.

5. Find the volume generated by the revolution of the entire curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, (a) about the X-axis; (b) about the Y-axis.

6. Find the volume generated by revolving about either axis the part of the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ intercepted between the axes.

7. A cylindrical vessel having an altitude of 12 inches and a base diameter of 8 inches is tipped and the contained fluid is poured out until the liquid surface coincides with a diameter of the base. Find the quantity of liquid remaining in the vessel.

8. If e is the eccentricity of an ellipse and ϕ the eccentric angle, the parametric equations of the ellipse are: $x = a \cos\phi$, $y = b \sin\phi$. Show that the entire length of the ellipse is

$$4a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \cos^2\phi}\, d\phi.$$

9. Find the volume generated by the revolution about OX of that part of the curve $y = e^x$ lying to the left of OY.

10. The solid shown in Fig. 78 is generated by moving a variable rectangle $DEFG$ parallel to the plane XOY. One angle D moves along the axis OZ, and the other angles E and G, move in given curves on the planes YZ and ZX, respectively. If the curves QER and PGR are circular arcs of radius a with centers at O, find the volume of the solid.

11. In Ex. 10 find the convex surface QSR.

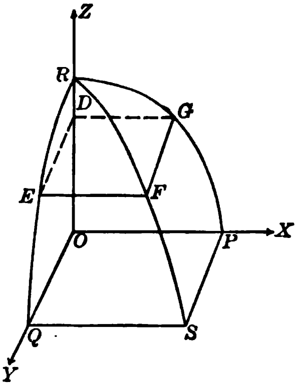

Fig. 78.

12. In Fig. 78, take $OR = 8$, $OP = OQ = 6$, and assume the curves RQ and RP to be parabolic arcs with vertices at R. Find the volume of the solid and the areas of the convex surfaces.

13. In Fig. 78, let the curves *RP* and *RQ* be elliptic quadrants with major and minor semi-axes of *m* and *n* respectively. Find the volume of the solid.

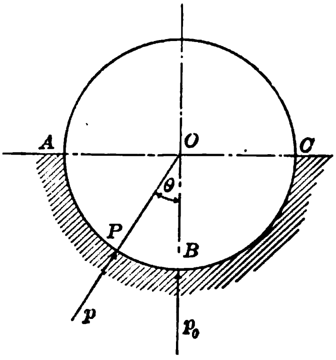

Fig. 79.

14. The value of a harmonic alternating current (Art. 60, b) is given by the equation $i = i_0 \sin \theta$, where i_0 is the maximum value. Find the mean value of the current for a half cycle, that is, from $\theta = 0$ to $\theta = \pi$.

15. In Fig. 79 is shown a cylindrical journal and bearing. The intensity of the normal pressure at any point *P* is assumed to be proportional to the depth of *P* below the diameter *AC*. If p_0 is the intensity at the lowest point *B*, find the mean intensity over the surface *ABC*.

16. Find an expression for the work done by a gas expanding isothermally according to van der Waals' equation

$$\left(p + \frac{a}{v^2}\right)(v - b) = C$$

from an initial volume v_1 to a final volume v_2.

17. Find an expression for the work done by a force which varies as the *n*th power of the displacement of its point of application.

18. Find the work required to compress 20 cubic feet of air from a pressure of 14.5 pounds per square inch to a pressure of 63.5 pounds per square inch, the equation of the compression curve being $pv^{1.3} = \text{const.}$

CHAPTER XI

SPECIAL METHODS OF INTEGRATION

94. Integration of rational fractions. Thus far we have made use of only those integrals which can be evaluated by use of the fundamental formulas given in Chapter VII. In some cases we were able to reduce the given function to a fundamental form by a simple transformation. Not all functions, however, can be easily reduced to those types by the methods already employed, and in this chapter we shall consider some special methods by which this reduction may be effected in cases more complicated than those already discussed.

In the present article, we shall show how the decomposition of rational fractions may be employed in simplifying an integration. By a rational fraction we mean the quotient of two rational integral functions. We shall consider only those fractions in which the numerator is of lower degree than the denominator; for, if this is not the case, we can always by division reduce the given expression to a rational integral function plus such a fraction. It is often convenient in integrating a rational fraction to express it as the sum of several fractions. The decomposition of rational fractions is fully treated in algebra,* and a knowledge of the principles and methods involved is assumed here. This method of integration can be applied whenever the denominator of the given fraction can be factored. We shall consider the following cases.

(a) *When the denominator is the product of several linear factors, none of which is repeated.*

The given function can then be written in the form

$$f(x) = \frac{\phi(x)}{(x - \alpha_1)\,(x - \alpha_2)\,\ldots\,(x - \alpha_k)}\,.$$

We may now assume

$$f(x) = \frac{A}{x - \alpha_1} + \frac{B}{x - \alpha_2} + \ldots + \frac{K}{x - \alpha_k},$$

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* See Fine's *College Algebra*, pp. 236–244.

and calculate the numerators $A, B, \ldots, K$ by the principles of undetermined coefficients. The integral of $f(x)$ is then found by taking the sum of the integrals of the separate terms. The following examples illustrate the method.

Ex. 1. $\displaystyle\int \frac{(5x+1)\,dx}{x^2-2x-35}$.

We have

$$\frac{5x+1}{x^2-2x-35} = \frac{5x+1}{(x+5)(x-7)} = \frac{A}{x+5} + \frac{B}{x-7}$$

Clearing of fractions, we obtain

$$5x+1 = A(x-7) + B(x+5).$$

Equating the coefficients of the same powers of x in the two members of this equation, there results

$$5 = A + B,$$
$$1 = 5B - 7A.$$

Solving for A and B, we obtain

$$A = 2, B = 3.$$

Substituting these values in the original equation, we have

$$\begin{aligned}\int \frac{(5x+1)\,dx}{(x+5)(x-7)} &= \int \frac{2\,dx}{x+5} + \int \frac{3\,dx}{x-7} \\ &= 2\log(x+5) + 3\log(x-7) + C \\ &= \log(x+5)^2(x-7)^3 + C.\end{aligned}$$

Ex. 2. $\displaystyle\int \frac{(x^2+x-1)\,dx}{x^3+x^2-6x}$.

Since $\qquad x^3 + x^2 - 6x = x(x+3)(x-2),$

we have, $\qquad \dfrac{x^2+x-1}{x(x+3)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$.

Clearing of fractions, we get the identical equation

$$x^2 + x - 1 = A(x+3)(x-2) + B(x-2)x + C(x+3)x.$$

For $x = 0$, we obtain $A = \frac{1}{6}$; for $x = -3$, $B = \frac{1}{3}$; and for $x = 2$. $C = \frac{1}{2}$. Hence we have

$$\frac{x^2+x-1}{x^3+x^2-6x} = \frac{1}{6x} + \frac{1}{3(x+3)} + \frac{1}{2(x-2)}.$$

Therefore,

$$\begin{aligned}\int \frac{(x^2+x-1)\,dx}{x^3+x^2-6x} &= \int \frac{dx}{6x} + \int \frac{dx}{3(x+3)} + \int \frac{dx}{2(x-2)} \\ &= \tfrac{1}{6}\log x + \tfrac{1}{3}\log(x+3) + \tfrac{1}{2}\log(x-2) + C \\ &= \log \sqrt[6]{x(x+3)^2(x-2)^3} + C.\end{aligned}$$

EXERCISES.

1. $\int \frac{(x+2)\,dx}{x^2-5x+6}$.
2. $\int \frac{(3x-5)\,dx}{x^2+2x-15}$.
3. $\int \frac{(3x+4)\,dx}{x^3-7x^2+12x}$.
4. $\int \frac{(2x-1)\,dx}{x^2+3x+2}$.
5. $\int \frac{dx}{2+7x+3x^2}$.
6. $\int \frac{x^4-5x^3+6x-4}{x^3-4x^2-5x}\,dx$.
7. $\int \frac{(x^2+mn)\,dx}{x(x+m)(x-n)}$.
8. $\int \frac{9x^2-4x-8}{x^3-4x}\,dx$. ✓

(b) *When the denominator is the product of several linear factors, some of which are repeated.*

In this case, the given function has the form

$$f(x) = \frac{\phi(x)}{(x-\alpha)\,(x-\beta)^k \dots (x-\gamma)^s}$$

which can be written in the form

$$f(x) = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{(x-\beta)^2} + \dots + \frac{K}{(x-\beta)^k} + \frac{L}{x-\gamma} + \dots + \frac{S}{(x-\gamma)^s}.$$

We may calculate the coefficients $A, B, C, \dots, K, L, \dots, S$, as in (a). The following example illustrates this method.

Ex. $\int \frac{(x-8)\,dx}{x^3-4x^2+4x}$.

We have

$$\frac{x-8}{x^3-4x^2+4x} = \frac{x-8}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Clearing of fractions, we get the identity

$$x - 8 = A(x-2)^2 + Bx(x-2) + Cx.$$

Equating the coefficients of like powers of x, we have

$$-8 = 4A, \qquad 1 = -4A - 2B + C, \qquad 0 = A + B;$$

and from these equations we obtain

$$A = -2, \qquad B = 2, \qquad C = -3.$$

Hence
$$\int \frac{(x-8)\,dx}{x^3-4x^2+4x} = -2\int\frac{dx}{x} + 2\int\frac{dx}{x-2} - 3\int\frac{dx}{(x-2)^2}$$

$$= -\log x^2 + \log(x-2)^2 + \frac{3}{x-2} + C$$

$$\log\left(\frac{x-2}{x}\right)^2 + \frac{3}{x-2} + C.$$

EXERCISES.

1. $\displaystyle\int \frac{(x-4)\,dx}{x^3-4x^2+4x}$.

2. $\displaystyle\int \frac{(3x-2)\,dx}{x(x+3)^2}$.

3. $\displaystyle\int \frac{(2x-5)\,dx}{(x-2)^3}$.

4. $\displaystyle\int \frac{(x-1)\,dx}{(x+3)^2}$.

5. $\displaystyle\int \frac{(4y-3)\,dy}{y^3-3y^2}$.

6. $\displaystyle\int \frac{6x^3-8x^2-4x+1}{x^4-2x^3+x^2}\,dx$.

7. $\displaystyle\int \frac{(3x-1)\,dx}{x^3-x^2-x+1}$.

8. $\displaystyle\int \frac{x\,dx}{(x+1)^2(x-1)}$.

9. $\displaystyle\int \frac{x^2+2x-3}{x^3-x^2}\,dx$.

10. $\displaystyle\int \frac{mx^2\,dx}{(m+x)^3}$.

(c) *When the denominator contains factors of the second degree.* Aside from linear factors, the denominator may contain quadratic factors not decomposable into real linear factors. For the linear factors, we assume a decomposition into partial fractions in accordance with the principles of cases (a) and (b). Corresponding to the quadratic factors, we assume fractions whose numerators are linear in the variable. If any of the quadratic factors appear to a degree higher than the first, then we assume in the decomposition as many fractions as the degree of the factor, the numerator of each being linear and the denominator having the given factor in increasing powers. The following examples illustrate the method.

Ex. 1. $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+1)}$.

We write
$$\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Clearing of fractions and equating coefficients of like powers of x, we find
$$A = B = C = \tfrac{1}{2}.$$

Hence
$$\int \frac{x^2\,dx}{(x-1)(x^2+1)} = \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx$$
$$= \frac{1}{2}\log(x-1) + \frac{1}{4}\log(x^2+1) + \frac{1}{2}\arctan x + C$$
$$= \frac{1}{4}\log(x-1)^2(x^2+1) + \frac{1}{2}\arctan x + C.$$

Ex. 2. $\displaystyle\int \frac{x+1}{(x-1)(x^2+1)^2}\,dx$

If we write
$$\frac{x+1}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2},$$

we find by clearing of fractions and equating like powers of x the following values for the coefficients:

$$A = \tfrac{1}{2},\ B = -\tfrac{1}{2},\ C = -\tfrac{1}{2},\ D = -1,\ E = 0.$$

Hence

$$\int \frac{x+1}{(x-1)(x^2+1)^2}dx = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x+1}{x^2+1}dx - \int \frac{x\,dx}{(x^2+1)^2}$$

$$= \frac{1}{2}\log(x-1) - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\arc\tan x + \frac{1}{2(x^2+1)} + C.$$

EXERCISES.

1. $\displaystyle\int \frac{5x^2 + 10x - 3}{x^4 - 1}\,dx.$

2. $\displaystyle\int \frac{dx}{x^4 - 1}.$

3. $\displaystyle\int \frac{(2x^4 - 1)\,dx}{x^2(1+x^2)^2}.$

4. $\displaystyle\int \frac{x\,dx}{x^4 + 4x^2 + 3}.$

5. $\displaystyle\int \frac{x\,dx}{x^4 + 5x^2 + 4}.$

6. $\displaystyle\int \frac{(z^2 - 1)\,dz}{z^4 + 2z^2 + 1}.$

7. $\displaystyle\int \frac{(y^2 - 4)\,dy}{y^4 + 2y^2 - 3}.$

8. $\displaystyle\int \frac{dx}{(x+a)(x^2+b^2)}.$

9. $\displaystyle\int \frac{dx}{x^2(x^2+3)}.$

10. $\displaystyle\int \frac{x\,dx}{x^3 + x^2 + x + 1}.$

11. $\displaystyle\int \frac{(2x^2 - 1)\,dx}{(1+x)^2(1+x+x^2)}.$

12. $\displaystyle\int \frac{dx}{x(1+2x^2)}.$

13. $\displaystyle\int \frac{x^3\,dx}{(x^2+1)^2}.$

95. Integration of functions containing radicals. A few irrational forms have been considered in the preceding articles. In the present article we shall consider some special methods by which such functions may frequently be changed into equivalent functions free from radicals.

(a) *When $f(x)$ contains fractional powers of $a + bx$, but no other radicals.*

In this case, $f(x)$ can be transformed into a rational expression by the substitution

$$a + bx = z^n, \tag{1}$$

where n is the least common multiple of the denominators of all the fractional exponents of $a + bx$. This follows from the fact that $f(x)$ and dx can then be expressed rationally in terms of z and dz, as the following examples will illustrate.

Ex. 1. Find $\int \frac{y\,dy}{(1+y)^{\frac{1}{3}}}$.

Put $$1 + y = z^3,$$

whence $$y = z^3 - 1,\quad dy = 3\,z^2\,dz.$$

We have then
$$\int \frac{y\,dy}{(1+y)^{\frac{1}{3}}} = \int \frac{(z^3-1)\,3\,z^2\,dz}{z} = 3\int (z^4 - z)\,dz$$
$$= 3\int z^4\,dz - 3\int z\,dz$$
$$= \tfrac{3}{5}\,z^5 - \tfrac{3}{2}\,z^2 + C$$
$$= \tfrac{3}{5}\,(y+1)^{\frac{5}{3}} - \tfrac{3}{2}\,(y+1)^{\frac{2}{3}} + C.$$

Ex. 2. Find $\int \frac{(x^{\frac{1}{2}} + x^{\frac{1}{4}} + 4)\,dx}{x^{\frac{1}{2}} + 1}$.

Put $x = z^4$, whence $dx = 4\,z^3\,dz$.
We have then

$$\int \frac{(x^{\frac{1}{2}} + x^{\frac{1}{4}} + 4)\,dx}{x^{\frac{1}{2}} + 1} = \int \frac{(z^2 + z + 4)\,4\,z^3\,dz}{z^2 + 1}$$
$$= 4\int \left(z^3 + z^2 + 3\,z - 1 - \frac{3\,z - 1}{z^2 + 1}\right) dz$$
$$= 4\left[\int z^3\,dz + \int z^2\,dz + 3\int z\,dz - \int dz - 3\int \frac{z\,dz}{z^2+1} + \int \frac{dz}{z^2+1}\right]$$
$$= 4\left[\tfrac{1}{4}\,z^4 + \tfrac{1}{3}\,z^3 + \tfrac{3}{2}\,z^2 - z - \tfrac{3}{2}\log(z^2+1) + \text{arc tan}\,z + C\right]$$
$$= z^4 + \tfrac{4}{3}\,z^3 + 6\,z^2 - 4\,z - 6\log(z^2+1) + 4\,\text{arc tan}\,z + C'$$
$$= x + \tfrac{4}{3}\,x^{\frac{3}{4}} + 6\,x^{\frac{1}{2}} - 4\,x^{\frac{1}{4}} - 6\log(x^{\frac{1}{2}} + 1) + 4\,\text{arc tan}\,x^{\frac{1}{4}} + C'.$$

(b) *When $f(x)$ contains the radical $\sqrt{x^2 + ax + b}$ and no other.* In this case, $f(x)\,dx$ can be rationalized by the substitution

$$\sqrt{x^2 + ax + b} = z - x. \qquad (2)$$

Squaring (2), we have

$$x^2 + ax + b = z^2 - 2\,zx + x^2,$$

whence $$x = \frac{z^2 - b}{a + 2\,z};$$

$$dx = \frac{2\,(z^2 + az + b)}{(a + 2\,z)^2}\,dz.$$

Consequently, $f(x)\,dx$ becomes free from radicals upon substituting these values of $\sqrt{x^2 + ax + b}$, x, and dx.

Ex. $\displaystyle\int\frac{x+\sqrt{x^2+x+1}}{2\sqrt{x^2+x+1}}\,dx.$

Put $\sqrt{x^2+x+1}=z-x,$

whence $$x=\frac{z^2-1}{2z+1},\quad dx=\frac{2\,(z^2+z+1)}{(2z+1)^2}\,dz.$$

Substituting these values, we have

$$\int\frac{x+\sqrt{x^2+x+1}}{2\sqrt{x^2+x+1}}\,dx=\int\frac{z\,dz}{2z+1}=\frac{1}{2}\int dz-\frac{1}{2}\int\frac{dz}{2z+1}$$
$$=\tfrac{1}{2}z-\tfrac{1}{4}\log(2z+1)+C.$$

Replacing the value of z, we obtain as the final result,

$$\tfrac{1}{2}\left[x+\sqrt{x^2+x+1}\right]-\tfrac{1}{4}\log\left[2x+2\sqrt{x^2+x+1}+1\right]+C.$$

(c) *When* $f(x)$ *contains the radical* $\sqrt{-x^2+ax+b}$ *and no other.*

We shall consider only those cases in which the expression under the radical can be broken up into real linear factors. The radical can therefore be written in the form

$$\sqrt{-x^2+ax+b}=\sqrt{(x-\alpha)(\beta-x)},$$

and $f(x)\,dx$ can be rationalized by the substitution

$$\sqrt{(x-\alpha)(\beta-x)}=(\beta-x)\,z\ [\text{or}\ \ (x-\alpha)\,z],$$

as we shall now show. Squaring, we obtain

$$(x-\alpha)(\beta-x)=(\beta-x)^2z^2,$$

whence $$x=\frac{\beta z^2+\alpha}{1+z^2},\quad dx=\frac{(\beta-\alpha)\,2z}{(1+z^2)^2}\,dz.$$

The integrand which results from the substitutions is rational.

Ex. Find $\displaystyle\int\frac{dx}{x\sqrt{-x^2+4x-3}}.$

We have here $\sqrt{(x-1)(3-x)}=(3-x)\,z,$

whence $$x=\frac{3z^2+1}{1+z^2},\quad dx=\frac{4z\,dz}{(1+z^2)^2}.$$

Substituting these values in the given integrand, we have

$$\int\frac{dx}{x\sqrt{-x^2+4x-3}}=2\int\frac{dz}{3z^2+1}=\frac{2}{\sqrt{3}}\arctan\sqrt{3}\,z+C$$
$$=\frac{2}{\sqrt{3}}\arctan\sqrt{\frac{3\,(x-1)}{3-x}}+C.$$

(d) *Integration by substituting* $x = \frac{1}{z}$.

In the cases considered so far in this section, the substitutions have been such as to rationalize the expression under the integral sign. It is often possible to perform the integration by means of a change of variable which does not rationalize the given function. There is no general law for such cases, but the substitution $x = \frac{1}{z}$ is often helpful.

Ex. 1. Find $\int \frac{dx}{x\sqrt{1 + x^2}}$.

Put $x = \frac{1}{z}$, whence $dx = -\frac{dz}{z^2}$.

We have then

$$\int \frac{dx}{x\sqrt{1 + x^2}} = -\int \frac{dz}{\sqrt{z^2 + 1}} = \log \frac{C}{z + \sqrt{z^2 + 1}}$$

$$= \log \frac{Cx}{1 + \sqrt{1 + x^2}}.$$

Ex. 2. Find $\int \frac{\sqrt{2\,ax - x^2}}{x^3}\,dx$.

Putting $x = \frac{1}{z}$, $dx = -\frac{dz}{z^2}$, we have

$$\int \frac{\sqrt{2\,ax - x^2}}{x^3}\,dx = -\int \sqrt{2\,az - 1}\,dz = -\frac{1}{3\,a}(2\,az - 1)^{\frac{3}{2}}$$

$$= -\frac{(2\,ax - x^2)^{\frac{3}{2}}}{3\,ax^3} + C.$$

EXERCISES.

1. $\int \frac{x}{1 + x^{\frac{1}{3}}}\,dx.$

2. $\int \frac{2\sqrt{x}\,dx}{4\sqrt{x} - 3\,x^{\frac{2}{3}}}.$

3. $\int x^2\,(mx + b)^{\frac{1}{2}}\,dx.$

4. $\int \frac{dx}{\sqrt{4\,x - 3 - x^2}}.$

5. $\int \frac{3\,dx}{\sqrt{7\,x - 10 - x^2}}.$

6. $\int \frac{dx}{\sqrt{x^2 + 5\,x - 3}}.$

7. $\int \frac{dx}{\sqrt{x^2 - 7\,x + 4}}.$

8. $\int \frac{dx}{x\sqrt{-x^2 + 5\,x - 6}}.$

9. $\int \frac{x^3\,dx}{\sqrt{x - 1}}.$

10. $\int \frac{(3\,x - 1)\,dx}{\sqrt{x^2 - 3}}.$

11. $\int \frac{dx}{\sqrt{x^2 + 2\,x + 5}}.$

12. $\int \frac{x^3\,dx}{\sqrt{1 - x^2}}.$

✓13. $\int \frac{x^3\,dx}{(1 + x^2)^{\frac{3}{2}}}.$

14. $\int \frac{\sqrt{2\,x - x^2}}{x^2}\,dx.$

15. $\int \frac{dx}{\sqrt{x^2 + x}}.$

16. $\int \frac{dx}{(x - 1)^{\frac{2}{3}} - (x - 1)^{\frac{1}{2}}}.$

17. Integrate Nos. 5, 6, 7, 11, 15 without rationalization.

Derive the following integrals:

18. $\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2}{15\,b^3}(8\,a^2 - 4\,abx + 3\,b^2x^2)\sqrt{a+bx}.$

19. $\int x\sqrt{a+bx}\,dx = -\frac{2}{15\,b^2}(2\,a - 3\,bx)(a+bx)^{\frac{3}{2}}.$

20. $\int x^2\sqrt{a+bx}\,dx = \frac{2}{105\,b^3}(8\,a^2 - 12\,abx + 15\,b^2x^2)(a+bx)^{\frac{3}{2}}.$

21. Derive Nos. 46, 47, 48 of the integrals for reference.

96. Integration by parts. If u and v are two functions of the same independent variable, we have upon differentiating their product,

$$d(uv) = u\,dv + v\,du,$$

or

$$u\,dv = d(uv) - v\,du.$$

Integrating, we get

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

By this formula the integral $\int u\,dv$ is obtained by the evaluation of another integral $\int v\,du$. This method is called *integration by parts*, and is one of the most useful of the integral calculus. It is particularly helpful in the integration of the product of two functions where the integral of one can be easily found; also in the integration of logarithmic functions, exponential functions, and inverse trigonometric functions. No general directions can be given as to which of the two functions is to be taken as u and which as dv. In general, we should make such a selection as will render dv and $v\,du$ most easily integrable. This can, however, be determined only by experience.

Ex. 1. $\int xe^x\,dx.$

Put $\quad dv = e^x dx, \quad u = x,$

whence $\quad v = e^x, \quad du = dx.$

Substituting these values in formula (1), we have

$$\int xe^x\,dx = xe^x - \int e^x\,dx = e^x(x-1) + C.$$

Ex. 2. $\int x \arctan x \, dx.$

Put $\qquad dv = x\,dx, \qquad u = \arctan x,$

whence $\qquad v = \frac{1}{2} x^2, \qquad du = \dfrac{dx}{1 + x^2}.$

Substituting these values in the formula, we have

$$\begin{aligned}\int x \arctan x \, dx &= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1 + x^2}\, dx \\ &= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{dx}{1 + x^2} \\ &= \frac{1}{2} x^2 \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + C \\ &= \frac{x^2 + 1}{2} \arctan x - \frac{1}{2} x + C.\end{aligned}$$

Ex. 3. $\int \log x \, dx.$

Put $\qquad dv = dx, \qquad u = \log x,$

whence $\qquad v = x, \qquad du = \dfrac{dx}{x}.$

Substituting these values of u, v, and du, we have

$$\begin{aligned}\int \log x \, dx = x \log x - \int \frac{x\,dx}{x} &= x \log x - x + C \\ &= x(\log x - 1) + C.\end{aligned}$$

Ex. 4. $\int \sqrt{a^2 - x^2}\, dx.$

Put $\qquad u = \sqrt{a^2 - x^2}, \qquad dv = dx,$

whence $\qquad du = -\dfrac{x\,dx}{\sqrt{a^2 - x^2}}, \quad v = x.$

The result of substituting these values in (1) is

$$\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + \int \frac{x^2\,dx}{\sqrt{a^2 - x^2}}.$$

We may write

$$\frac{x^2}{\sqrt{a^2 - x^2}} = -\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} + \frac{a^2}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} + \frac{a^2}{\sqrt{a^2 - x^2}}.$$

We have therefore

$$\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}},$$

whence $$2\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a},$$

and $$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right].$$

EXERCISES.

1. $\int x^2e^x\,dx.$
2. $\int x^m \log x\,dx.$
3. $\int \text{arc sin}\,\theta\,d\theta.$
4. $\int \text{arc cot}\,\theta\,d\theta.$
5. $\int \text{inv. cosh}\,x\,dx.$
6. $\int e^{ax}\sin bx\,dx.$
7. $\int_0^1 \text{inv. sinh}\,x\,dx.$
8. $\int \cos\theta\log\sin\theta\,d\theta.$
9. $\int x^2\,\text{arc tan}\,x\,dx.$
10. $\int e^{\frac{x}{a}}\cos\frac{x}{a}\,dx.$
11. $\int x\cos x\,dx.$
12. $\int \sec^3\theta\,d\theta.$
13. $\int_0^a \sqrt{x^2+a^2}\,dx.$
14. $\int_0^a x^3\sqrt{a^2-x^2}\,dx.$
15. $\int x^2\log^2 x\,dx.$
16. $\int_0^{\frac{\pi}{2}} \cos\theta\cos 2\,\theta\,d\theta.$

97. Reduction formulas for algebraic functions. As has been seen, we may use the method of integration by parts to make certain integrals depend upon others of a simpler character. To apply this method to all such cases is often a laborious task. Therefore, we shall now develop certain reduction formulas which may be used to advantage in the more difficult cases. The student is advised to use these formulas only for reference; they need not be memorized.

The general form of the integral to which these reduction formulas apply is

$$\int x^m\,(a+bx^n)^p\,dx,$$

where a, b, m, n, p are constants. The object of the reduction formula is to increase or decrease either m or p in such a way as to make the integration depend finally upon the evaluation of some known integral. The four principal formulas are the following:

$$\int x^m(a+bx^n)^p\,dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{(np+m+1)\,b} - \frac{(m-n+1)\,a}{(np+m+1)\,b}\int x^{m-n}(a+bx^n)^p\,dx. \tag{A}$$

$$\int x^m(a+bx^n)^p\,dx = \frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}\,dx. \tag{B}$$

$$\int x^m(a+bx^n)^p\,dx = \frac{x^{m+1}(a+bx^n)^{p+1}}{(m+1)\,a} - \frac{(np+n+m+1)\,b}{(m+1)\,a}\int x^{m+n}(a+bx^n)^p\,dx. \tag{C}$$

$$\int x^m(a+bx^n)^p\,dx = -\frac{x^{m+1}(a+bx^n)^{p+1}}{an\,(p+1)} + \frac{np+n+m+1}{an\,(p+1)}\int x^m(a+bx^n)^{p+1}\,dx. \tag{D}$$

Formula (A) reduces m by n, (B) decreases p by 1, (C) increases m by n, and (D) increases p by 1. Evidently (A) and (B) are not applicable if $np + m + 1 = 0$, (C) fails for $m = -1$, and (D) fails for $p = -1$.

The proof of formula (A) is as follows:

In the formula for integration by parts,

$$\int u\,dv = uv - \int v\,du,$$

put $\quad u = x^{m-n+1}, \qquad dv = (a+bx^n)^p\,x^{n-1}\,dx,$

whence $\quad du = (m-n+1)\,x^{m-n}\,dx, \qquad v = \dfrac{(a+bx^n)^{p+1}}{nb\,(p+1)}\,.$

We have then

$$\int x^m(a+bx^n)^p\,dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb\,(p+1)} - \frac{m-n+1}{nb\,(p+1)}\int x^{m-n}(a+bx^n)^{p+1}\,dx. \tag{1}$$

But

$$\int x^{m-n}(a + bx^n)^{p+1}\,dx = a\int x^{m-n}(a + bx^n)^p\,dx + b\int x^m(a + bx^n)^p\,dx. \qquad (2)$$

Substituting this expression for the integral in the second member of (1), we have after transposing and combining,

$$\int x^m(a + bx^n)^p\,dx = \frac{x^{m-n+1}(a + bx^n)^{p+1}}{(np + m + 1)\,b} - \frac{(m - n + 1)\,a}{(np + m + 1)\,b}\int x^{m-n}(a + bx^n)^p\,dx,$$

which is the required formula.

To derive formula (B), write the integral in the form

$$\begin{aligned}\int x^m(a + bx^n)^p\,dx &= \int x^m(a + bx^n)^{p-1}(a + bx^n)\,dx\\ &= a\int x^m(a + bx^n)^{p-1}\,dx + b\int x^{m+n}(a + bx^n)^{p-1}\,dx, \qquad (3)\end{aligned}$$

and apply formula (A) to the last of these integrals. The result of applying (A) is

$$\int x^{m+n}(a + bx^n)^{p-1}\,dx = \frac{x^{m+1}(a + bx^n)^p}{b(np + m + 1)} - \frac{a(m + 1)}{b(np + m + 1)}\int x^m(a + bx^n)^{p-1}\,dx.$$

Substituting this expression in (3), the result is

$$\int x^m(a + bx^n)^p\,dx = \frac{x^{m+1}(a + bx^n)^p}{np + m + 1} + \frac{anp}{np + m + 1}\int x^m(a + bx^n)^{p-1}\,dx.$$

Formula (C) may be derived directly from formula (A). Solving (A) for $\int x^{m-n}(a + bx^n)^p\,dx$, we obtain

$$\int x^{m-n}(a + bx^n)^p\,dx = \frac{x^{m-n+1}(a + bx^n)^{p+1}}{(m - n + 1)\,a} - \frac{(np + m + 1)\,b}{(m - n + 1)\,a}\int x^m(a + bx^n)^p\,dx,$$

and this takes the required form upon the substitution of $m + n$ for m. Likewise (D) is derived from (B) by solving (B) for $\int x^m (a + bx^n)^{p-1}\,dx$ and substituting $p + 1$ for p.

Ex. 1. $\int \frac{x^2\,dx}{\sqrt{c^2 - x^2}}$.

If we write the integral in the form $\int x^2 (c^2 - x^2)^{-\frac{1}{2}}\,dx$, we see that the reduction of the exponent m by n will give a fundamental form. Hence formula (A) applies.

We have then

$$m = 2,\ n = 2,\ p = -\tfrac{1}{2},\ a = c^2,\ b = -1.$$

Substituting these values in formula (A), the result is

$$\int \frac{x^2\,dx}{\sqrt{c^2 - x^2}} = \frac{x (c^2 - x^2)^{\frac{1}{2}}}{-2} - \frac{c^2}{-2}\int (c^2 - x^2)^{-\frac{1}{2}}\,dx$$

$$= -\frac{x}{2}\sqrt{c^2 - x^2} + \frac{c^2}{2}\text{arc sin}\,\frac{x}{c} + C.$$

Ex. 2. $\int \sqrt{c^2 + x^2}\,dx.$

This integral may be made to depend upon the known integral $\int \frac{dx}{\sqrt{c^2 + x^2}}$ by the application of formula (B). We have in this case,

$$m = 0,\ n = 2,\ p = \tfrac{1}{2},\ a = c^2,\ b = 1.$$

Hence by the formula, we have

$$\int (c^2 + x^2)^{\frac{1}{2}}\,dx = \frac{x}{2}(c^2 + x^2)^{\frac{1}{2}} + \frac{c^2}{2}\int \frac{dx}{\sqrt{c^2 + x^2}}$$

$$= \frac{x}{2}\sqrt{c^2 + x^2} + \frac{c^2}{2}\log\left(x + \sqrt{c^2 + x^2}\right) + C.$$

Ex. 3. $\int \frac{dx}{x^3\sqrt{x^2 - c^2}}$.

We have $\quad m = -3,\ n = 2,\ p = -\tfrac{1}{2},\ a = -c^2,\ b = 1.$

Substituting these values in (C), the result is

$$\int \frac{dx}{x^3\sqrt{x^2 - c^2}} = \frac{x^{-2}(x^2 - c^2)^{\frac{1}{2}}}{2c^2} + \frac{1}{2c^2}\int x^{-1}(x^2 - c^2)^{-\frac{1}{2}}\,dx$$

$$= \frac{\sqrt{x^2 - c^2}}{2c^2x^2} + \frac{1}{2c^3}\text{arc sec}\,\frac{x}{c} + C.$$

Ex. 4. $\int \frac{x^2\,dx}{(c^2 - x^2)^{\frac{3}{2}}}$.

Here we have $\quad m = 2,\ n = 2,\ p = -\tfrac{3}{2},\ a = c^2,\ b = -1.$

Substituting these values in (D), we obtain

$$\int \frac{x^2\,dx}{(c^2 - x^2)^{\frac{3}{2}}} = \frac{x^3}{c^2(c^2 - x^2)^{\frac{1}{2}}} - \int \frac{2}{c^2}\frac{x^2\,dx}{\sqrt{c^2 - x^2}}.$$

From Ex. 1, we have

$$\int \frac{x^2\,dx}{\sqrt{c^2 - x^2}} = -\frac{x}{2}\sqrt{c^2 - x^2} + \frac{c^2}{2}\text{arc sin}\frac{x}{c} + C.$$

Hence the final result is

$$\int \frac{x^2\,dx}{(c^2 - x^2)^{\frac{3}{2}}} = \frac{x^3}{c^2\sqrt{c^2 - x^2}} + \frac{x}{c^2}\sqrt{c^2 - x^2} - \text{arc sin}\frac{x}{c} + C$$

$$= \frac{x}{\sqrt{c^2 - x^2}} - \text{arc sin}\frac{x}{c} + C.$$

EXERCISES.

1. $\int \frac{x^4\,dx}{\sqrt{1 - x^2}}.$
2. $\int \frac{x^3\,dx}{\sqrt{1 + x^2}}.$
3. $\int_0^a \frac{dx}{(a^2 + x^2)^3}.$
4. $\int \sqrt{(a^2 - x^2)^3}\,dx.$
5. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}}.$
6. $\int_0^a x^2\sqrt{a^2 - x^2}\,dx.$
7. $\int \frac{dx}{x^4\sqrt{x^2 - 1}}.$
8. $\int \frac{x^2\,dx}{\sqrt{2ax - x^2}}.$
9. $\int \frac{(x - a)\,dx}{x^3\sqrt{x^2 - a^2}}.$
10. $\int x^{-3}\sqrt{2ax - x^2}\,dx.$
11. $\int \frac{x^2\,dx}{(a + bx^2)^3}.$
12. $\int \frac{x^4\,dx}{(x^2 + a^2)^{\frac{3}{2}}}.$

13. By the methods of this article derive two or more of the reduction formulas given in the integrals for reference, Nos. 83 to 86.

98. Integration of special trigonometric functions. There are certain types of trigonometric functions, which are readily integrated by easy reductions to standard forms. The following are of this kind:

(a) $$\int \sec^{2n} x\,dx, \qquad \int \csc^{2n} x\,dx.$$

Here n is assumed to be a positive integer. By the substitution $\sec^2 x = 1 + \tan^2 x$, $\sec^{2n} x$ becomes $(1 + \tan^2 x)^{n-1}\sec^2 x$; hence

we have $$\int \sec^{2n} x\,dx = \int (1 + \tan^2 x)^{n-1}\,d\,(\tan x).$$

Similarly, $$\int \csc^{2n} x\,dx = -\int (1 + \cot^2 x)^{n-1}\,d\,(\cot x).$$

In each case the binomial can be expanded, and the separate terms of the integrand are then standard forms.

Ex. 1. $$\begin{aligned}\int \sec^4 x\,dx &= \int (1 + \tan^2 x) \sec^2 x\,dx \\ &= \int \sec^2 x\,dx + \int \tan^2 x \sec^2 x\,dx \\ &= \tan x + \tfrac{1}{3} \tan^3 x + C.\end{aligned}$$

Evidently integrals of the form

$$\int \tan^m x \sec^{2n} x\,dx, \qquad \int \cot^m x \sec^{2n} x\,dx$$

can be integrated equally well by this device.

(b) $$\int \sec^m x \tan^{2n+1} x\,dx, \qquad \int \csc^m x \cot^{2n+1} x\,dx.$$

Here n is a positive integer (or zero), and m is any number. We have

$$\begin{aligned}\int \sec^m x \tan^{2n+1} x\,dx &= \int \sec^{m-1} x \cdot (\sec^2 x - 1)^n \sec x \tan x\,dx \\ &= \int \sec^{m-1} x \cdot (\sec^2 x - 1)^n\,d\,(\sec x),\end{aligned}$$

a form which is readily integrated. A similar form may be deduced for the second integral.

Ex. 2. $$\begin{aligned}\int \sec^4 x \tan^3 x\,dx &= \int \sec^3 x \tan^2 x \tan x \sec x\,dx \\ &= \int \sec^3 x\,(\sec^2 x - 1)\,d\,(\sec x) \\ &= \int \sec^5 x\,d\,(\sec x) - \int \sec^3 x\,d\,(\sec x) \\ &= \tfrac{1}{6} \sec^6 x - \tfrac{1}{4} \sec^4 x + C.\end{aligned}$$

(c) $$\int \sin^{2n+1} x\,dx, \qquad \int \cos^{2n+1} x\,dx.$$

Again n is a positive integer. We have in this case,

$$\int \sin^{2n+1} x\,dx = \int (1 - \cos^2 x)^n \sin x\,dx = -\int (1 - \cos^2 x)^n\,d\,(\cos x).$$

$$\int \cos^{2n+1} x\,dx = \int (1 - \sin^2 x)^n \cos x\,dx = \int (1 - \sin^2 x)^n\,d\,(\sin x).$$

Either of these may now be integrated by expanding the parenthesis and integrating the result term by term.

Ex. 3. $$\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx$$
$$= -\int (1 - \cos^2 x)^2\,d(\cos x)$$
$$= -\int d(\cos x) + 2\int \cos^2 x\,d(\cos x) - \int \cos^4 x\,d(\cos x)$$
$$= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C.$$

(d) $$\int \sin^{2n} x\,dx, \qquad \int \cos^{2n} x\,dx.$$

We may perform the integration in this case by the use of multiple angles. From trigonometry we have

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x),$$
$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$

By the aid of these formulas the integrand can be expressed in terms of multiple angles of the variable, where the sine and cosine appear only to the first degree.

When the degree of the original function is high, this process is somewhat cumbersome, and for such cases, it is better to make use of a reduction formula.

Ex. 4. $\int \cos^4 x\,dx.$

We may write this as follows:

$$\int (\cos^2 x)^2\,dx = \tfrac{1}{4}\int (1 + \cos 2x)^2\,dx = \tfrac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$
$$= \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}\int (1 + \cos 4x)\,dx$$
$$= \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}x + \tfrac{1}{32}\sin 4x + C$$
$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$$

(e) $$\int \sin^m x \cos^n x\,dx.$$

In general, the integration of this form is effected by the reduction formulas given in the next article. There are, however, three cases in which the relation between m and n may lead to simpler integrations than the general method.

(1) *When either m or n is a positive odd integer.*

In this case, we can reduce the integral to the fundamental form $\int u^n du$. For example, suppose n is odd, say equal to $2k + 1$. The given integral may then be written

$$\int \cos^m x \sin^{2k} x \sin x \, dx = - \int \cos^m x \, (1 - \cos^2 x)^k \, d \, (\cos x),$$

which may be integrated easily by expanding the parenthesis and integrating the result term by term.

Ex. 5. $\int \cos^5 x \sin^3 x \, dx.$

This may be written in the form

$$\int \cos^5 x \, (1 - \cos^2 x) \sin x \, dx = \int \cos^5 x \sin x \, dx - \int \cos^7 x \sin x \, dx$$

$$= -\tfrac{1}{6} \cos^6 x + \tfrac{1}{8} \cos^8 x + C.$$

(2) When *$m + n$ is a negative even integer.*

In this case we may write

$$\int \sin^m x \cos^n x \, dx = \int \frac{\sin^m x}{\cos^m x} \cos^{m+n} x \, dx$$

$$= \int \tan^m x \sec^{-(m+n)} x \, dx.$$

Since $-(m + n)$ is by hypothesis a positive even integer, the integral falls under type (a).

Ex. 6. $\int \frac{\sin^2 x}{\cos^4 x} dx = \int \tan^2 x \sec^2 x \, dx = \tfrac{1}{3} \tan^3 x + C.$

(3) *When m and n are positive and both even.*

We may then reduce the given integral to an integral depending upon the integration of the sine of some multiple of the given angle. This may be accomplished by the aid of the substitution

$$\sin 2x = 2 \sin x \cos x$$

and those given in (d).

EXERCISES.

1. $\int \sec^6 x \tan x \, ax.$
2. $\int \csc^4 x \, dx.$
3. $\int \cot^2 x \csc^4 x \, dx.$
4. $\int \frac{\cos^2 x}{\sin^4 x} dx.$
5. $\int \sin^3 x \cos^2 x \, dx.$
6. $\int \frac{dx}{\sin^3 x \cos x}.$
7. $\int \tan^3 x \, dx.$
8. $\int \sec^2 x \tan^5 x \, dx.$
9. $\int \sin x \cot^3 x \, dx.$
10. $\int \frac{\sqrt{\sin x}}{\cos^{\frac{5}{2}} x} dx.$
11. $\int \sin^{\frac{1}{3}} x \cos^3 x \, dx.$
12. $\int \cot^2 x \sec^4 x \, dx.$

99. Reduction formulas for trigonometric integrals. In many cases, integration of the forms

$$\int \sin^n x \, dx, \quad \int \cos^n x \, dx, \quad \int \sin^m x \cos^n x \, dx$$

is most easily effected by the use of general reduction formulas derived from the method of integration by parts.

For $\int \sin^n x \, dx$ we proceed as follows:

Put $\qquad u = \sin^{n-1} x, \qquad dv = \sin x \, dx,$

$\qquad du = (n-1) \sin^{n-2} x \cos x \, dx, \quad v = -\cos x.$

We may then write

$$\begin{aligned}\int \sin^n x \, dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx \\ &\qquad - (n-1) \int \sin^n x \, dx.\end{aligned}$$

Transposing the last term of the second member to the first member, we have, after combining and dividing by n,

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx. \qquad \text{(A)}$$

Successive application of this reduction formula will make the given integral depend upon $\int dx$ or $\int \sin x\, dx$, according as n is even or odd.

If we consider the integral $\int \cos^n x\, dx$, we may deduce in a similar manner the reduction formula

$$\int \cos^n x\, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx. \tag{B}$$

For the integral $\int \sin^m x \cos^n x\, dx$ we have the following reduction formulas.

$$\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\, dx. \tag{C}$$

$$\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx. \tag{D}$$

cf A

when n = 0

$$\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x\, dx. \tag{E}$$

$$\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x\, dx. \tag{F}$$

To develop formula (C), we put

$$u = \cos^{n-1} x, \qquad dv = \sin^m x \cos x\, dx,$$

$$du = -(n-1)\cos^{n-2} x \sin x\, dx, \qquad v = \frac{\sin^{m+1} x}{m+1}.$$

Substituting these values, we obtain

$$\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x\, dx. \tag{1}$$

But we have

$$\int \sin^{m+2} x \cos^{n-2} x\, dx = \int \sin^m x\,(1 - \cos^2 x) \cos^{n-2} x\, dx$$

$$= \int \sin^m x \cos^{n-2} x\, dx - \int \sin^m x \cos^n x\, dx. \tag{2}$$

Substituting this in (1), we have

$$\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x\, dx$$

$$- \frac{n-1}{m+1} \int \sin^m x \cos^n x\, dx. \tag{3}$$

Transposing the last integral of the second member to the first member, we have the given formula, when we solve the resulting equation for $\int \sin^m x \cos^n x\, dx$.

Formula (C) is useful when we wish to decrease the value of n, keeping the value of m fixed. It holds for all values of m and n except such as make $m + n = 0$.

Formula (D) can be derived in a similar manner from the formula for the integration by parts, if we put

$$u = \sin^{m-1} x, \qquad dv = \cos^n x \sin x\, dx,$$

$$du = (m-1) \sin^{m-2} x \cos x\, dx, \qquad v = -\frac{\cos^{n+1} x}{n+1}.$$

It is useful when we wish to decrease the value of m, keeping the value of n fixed. This formula also fails for $m + n = 0$.

When we wish to increase the value of n, keeping that of m unchanged, formula (E) is used. (E) may be derived from (C) by first solving the equation for $\int \sin^m x \cos^{n-2} x\, dx$ and then replacing n by $n + 2$. This formula does not hold for $n = -1$.

Formula (F) is used to increase the value of m and leave the value of n unchanged. It is valid in all cases save those in which $m + 1 = 0$. This formula can be obtained directly from (D) by solving that equation for $\int \sin^{m-2} x \cos^n x\, dx$ and then replacing m by $m + 2$.

The use of the reduction formulas is illustrated by the following examples.

Ex. 1. $\int \sin^6 x\, dx.$

Apply the reduction formula (A). We have, since $n = 6$,

$$\int \sin^6 x\, dx = -\frac{\sin^5 x \cos x}{6} + \tfrac{5}{6}\int \sin^4 x\, dx. \tag{1}$$

Applying the same formula to $\int \sin^4 x\, dx$, we have

$$\int \sin^4 x\, dx = -\frac{\sin^3 x \cos x}{4} + \tfrac{3}{4}\int \sin^2 x\, dx, \tag{2}$$

where, also by (A),

$$\int \sin^2 x\, dx = -\frac{\sin x \cos x}{2} + \tfrac{1}{2}\int dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2} x + C. \tag{3}$$

Substituting in (1) the results given in (2) and (3), we have

$$\int \sin^6 x\, dx = -\tfrac{1}{6}\sin^5 x \cos x - \tfrac{5}{24}\sin^3 x \cos x - \tfrac{15}{48}(\sin x \cos x - x) + C.$$

Ex. 2. $\int \sin^4 x \cos^2 x\, dx.$

Applying formula (D), we have

$$\int \sin^4 x \cos^2 x\, dx = -\frac{\sin^3 x \cos^3 x}{6} + \tfrac{1}{2}\int \sin^2 x \cos^2 x\, dx. \tag{1}$$

But

$$\int \sin^2 x \cos^2 x\, dx = \tfrac{1}{4}\int \sin^2 2x\, dx = \tfrac{1}{8}\int (1 - \cos 4x)\, dx$$

$$= \frac{x}{8} - \tfrac{1}{32}\sin 4x + C. \tag{2}$$

Substituting this value in (1), we obtain

$$\int \sin^4 x \cos^2 x\, dx = -\tfrac{1}{6}\sin^3 x \cos x + \tfrac{1}{16} x - \tfrac{1}{64}\sin 4x + C.$$

EXERCISES.

1. $\int \sin^3 x\, dx.$
2. $\int \cos^5 x\, dx.$
3. $\int \sin^2 \theta\, d\theta.$
4. $\int \sin^4 \theta\, d\theta.$
5. $\int \cos^6 x\, dx.$
6. $\int \sin^2 x \cos^3 x\, dx.$
7. $\int \sin^5 x \cos^2 x\, dx.$
8. $\int \sin^3 \theta \cos^3 \theta\, d\theta.$
9. $\int \frac{dx}{\sin^{\frac{5}{2}} x \cos^{\frac{3}{2}} x}.$
10. $\int \frac{\sin^{\frac{3}{2}} x\, dx}{\cos^{\frac{1}{2}} x}.$

11. $\int \frac{\sin^2 x\, dx}{\cos^3 x}$.

12. $\int \sin^3 x \cos^{-2} x\, dx$.

13. $\int \frac{dx}{\sin x \cos^3 x}$.

14. $\int \frac{dx}{\sin^3 x}$.

15. $\int \frac{dx}{\cos^4 x}$.

16. $\int \frac{dx}{\sin^3 x \cos x}$.

17. $\int \cot^3 x\, dx$.

18. $\int \sec^3 x\, dx$.

19. $\int \frac{d\theta}{\sin^2 \theta \cos^2 \theta}$.

20. $\int \sin^4 \theta \cos^6 \theta\, d\theta$.

100. Integration by trigonometric substitution. If $f(x)$ contains a radical of the form $\sqrt{a^2 \pm u^2}$, where u is some function of x, it can often be easily transformed to an equivalent function free from radicals by the substitution of a trigonometric function. All that is necessary is to substitute for u that trigonometric function which renders the expression under the radical a perfect square. It will be readily seen that this end is accomplished by the following substitutions:

(1) Put $u = a \sin \theta$, or $a \cos \theta$ in functions involving $\sqrt{a^2 - u^2}$.

(2) Put $u = a \tan \theta$, or $a \cot \theta$ in functions involving $\sqrt{a^2 + u^2}$.

(3) Put $u = a \sec \theta$, or $a \csc \theta$ in functions involving $\sqrt{u^2 - a^2}$.

Whenever the resulting trigonometric function does not fall at once under one of the fundamental formulas, the student should apply the methods of integration discussed in the preceding articles.

Frequently the substitution of a hyperbolic function for the variable leads to an easy integration. If the integral contains $\sqrt{a^2 + x^2}$ substitute $x = a \sinh u$, if $\sqrt{a^2 - x^2}$ substitute $a \cosh u$, etc.

The following examples illustrate the use of these substitutions.

Ex. 1. $\int \sqrt{a^2 - x^2}\, dx$.

Put $\qquad x = a \sin z,\ dx = a \cos z\, dz.$

We have then

$$\begin{aligned}\int \sqrt{a^2 - x^2}\, dx &= a^2 \int \cos^2 z\, dz = \tfrac{1}{2} a^2 \int (1 + \cos 2z)\, dz\\ &= \tfrac{1}{2} a^2 \int dz + \tfrac{1}{4} a^2 \int \cos 2z\, d(2z)\\ &= \tfrac{1}{2} a^2 z + \tfrac{1}{4} a^2 \sin 2z + C\\ &= \tfrac{1}{2} a^2 \arcsin \frac{x}{a} + \tfrac{1}{2} x \sqrt{a^2 - x^2} + C.\end{aligned}$$

Ex. 2. $\displaystyle\int\frac{dx}{x^2(x^2-a^2)^{\frac{1}{2}}}$.

Put $x = a\sec z$, $dx = a\sec z\tan z\,dz$.

Then we have

$$\int\frac{dx}{x^2(x^2-a^2)^{\frac{1}{2}}} = \int\frac{a\sec z\tan z\,dz}{a^2\sec^2 z(a^2\sec^2 z-a^2)^{\frac{1}{2}}}$$

$$= \frac{1}{a^2}\int\cos z\,dz = \frac{\sin z}{a^2} + C = \frac{\sqrt{x^2-a^2}}{a^2x} + C.$$

Ex. 3. Find $\displaystyle\int\sqrt{x^2+a^2}\,dx.$

Let $x = a\sinh u$, hence $dx = a\cosh u\,du$, and

$$\int\sqrt{x^2+a^2}\,dx = a^2\int\cosh^2 u\,du = \tfrac{1}{2}a^2(u+\sinh u\cosh u).$$

Since $\sinh u = \dfrac{x}{a}$, $\cosh u = \sqrt{1+\dfrac{x^2}{a^2}}$, and $u = \text{inv.}\sinh\dfrac{x}{a}$.

Hence $$\int\sqrt{x^2+a^2}\,dx = \frac{a^2}{2}\text{inv.}\sinh\frac{x}{a} + \frac{x}{2}\sqrt{x^2+a^2}$$

$$= \frac{a^2}{2}\log\frac{x+\sqrt{x^2+a^2}}{a} + \frac{x}{2}\sqrt{x^2+a^2} + C.$$

EXERCISES.

1. $\displaystyle\int\frac{dx}{x^2\sqrt{a^2-x^2}}$.

2. $\displaystyle\int\frac{x^2\,dx}{(a^2-x^2)^{\frac{3}{2}}}$.

3. $\displaystyle\int\frac{x^2\,dx}{(x^2-a^2)^{\frac{3}{2}}}$.

4. $\displaystyle\int\frac{x^2\,dx}{\sqrt{x^2-a^2}}$.

5. $\displaystyle\int(x^2-a^2)^{\frac{3}{2}}\,dx.$

6. $\displaystyle\int\frac{x^4\,dx}{\sqrt{1-x^2}}$.

7. $\displaystyle\int\frac{x^3\,dx}{\sqrt{1+x^2}}$.

8. $\displaystyle\int\frac{dx}{x^4\sqrt{x^2-1}}$

9. $\displaystyle\int_0^a\sqrt{2ax-x^2}\,dx.$ [Put $x = a(1+\sin\theta)$.]

10. $\displaystyle\int_1^4\frac{dx}{x^4\sqrt{x^2-1}}$. [Put $x = \sec\theta$.]

11. Show that by the substitution $x = a\tan\theta$ the integral $\displaystyle\int\frac{dx}{(a^2+x^2)^n}$ is reduced to an integral of the form $\displaystyle\int\cos^p\theta\,d\theta.$

MISCELLANEOUS EXERCISES.

1. Integrate the following:

(a) $\int \frac{dx}{x^2(1 + x^2)^{\frac{3}{2}}}$. (b) $\int \frac{dx}{\sqrt{4 + 3x - 2x^2}}$. (c) $\int \frac{\sqrt[3]{x} - \sqrt{x}}{4x^{\frac{1}{2}}} dx$.

(d) $\int \frac{dx}{\sqrt{a+\sqrt{x}}}$. (e) $\int \frac{x^3\, dx}{\sqrt{2x^2 + 1}}$. (f) $\int \frac{\cos\theta\, d\theta}{\cos(\theta + \epsilon)}$.

2. Integrate in three different ways each of the following:

(a) $\int (a^2 - x^2)^{\frac{3}{2}} dx$. (b) $\int x^2 \sqrt{a^2 - x^2}\, dx$.

3. Show that $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n - 1} - \int \tan^{n-2} x\, dx$.

Suggestion: Use the substitution $\tan^2 x = \sec^2 x - 1$.

4. Making use of the reduction formula of Ex. 3, find

(a) $\int \tan^5 x\, dx$. (b) $\int \tan^4 \theta\, d\theta$.

5. Perform the following integrations:

(a) $\int \frac{3x - 2}{x^2 - 5x + 4} dx$. (b) $\int x^2 \operatorname{arc\,cos} x\, dx$.

(c) $\int e^{\operatorname{arc\,tan} x} \frac{dx}{(1 + x^2)^{\frac{3}{2}}}$. (d) $\int e^{2x} \cos^2 x\, dx$.

6. Integrate $\int \frac{x\sqrt{a^2 - x^2}}{\sqrt{a^2 + x^2}} dx$ by the substitution $x^2 = a^2 \cos 2\theta$.

7. Prove that $\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$

$$= \frac{1 \cdot 3 \cdot 5 \ldots (n-1)}{2 \cdot 4 \cdot 6 \ldots n} \cdot \frac{\pi}{2}, \text{ if } n \text{ is an even integer,}$$

$$= \frac{2 \cdot 4 \cdot 6 \ldots (n-1)}{1 \cdot 3 \cdot 5 \ldots n}, \text{ if } n \text{ is an odd integer.}$$

8. Evaluate the following definite integrals:

(a) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta \sin\theta \cos\theta d\theta$. (b) $\int_0^{\pi} \theta \sin^2\theta \cos\theta\, d\theta$. (c) $\int_0^{\frac{\pi}{2}} \frac{\sin 4\theta}{\sin\theta} d\theta$.

9. Integrate the following:

(a) $\int \frac{dx}{(1 - x^2)\sqrt{1 + x^2}}$. (b) $\int \frac{x dx}{(x^2 + a^2)(x^2 + b^2)}$. (c) $\int x^3 \sqrt{a + bx^2} dx$.

10. By means of the substitution $z = \tan\frac{x}{2}$ verify the following important results:

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \text{ arc tan}\left[\sqrt{\frac{a - b}{a + b}} \tan\frac{x}{2}\right], \ (a^2 > b^2)$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b + a} + \sqrt{b - a}\tan\frac{x}{2}}{\sqrt{b + a} - \sqrt{b - a}\tan\frac{x}{2}}, \ (a^2 < b^2).$$

$$\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \text{ arc tan}\left[\frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}}\right], \ (a^2 > b^2)$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{a\tan\frac{x}{2} + b - \sqrt{b^2 - a^2}}{a\tan\frac{x}{2} + b + \sqrt{b^2 - a^2}}, \ (a^2 < b^2).$$

11. The speed of a certain type of chemical reaction is given by the equation

$$\frac{dx}{dt} = k(a - x)(b - x)(c - x),$$

in which k is a constant and a, b, and c denote the concentration of the substances (Mellor, *Higher Math.* p. 218). (a) Derive an expression for k, knowing that $x = 0$ when $t = 0$. (b) Find an expression for x when $a = b = c$.

12. The effective value of a harmonic electric current is that which gives the same heating effect as the actual varying current. Since the heating effect is proportional to the square of the current, we have

$$(\text{effective current})^2 = \text{mean value of (actual current)}.^2$$

Hence denoting the effective current by i_e, we have (see Article 60 (b)).

$$i_e^2 = \frac{\int_0^\pi (i_0 \sin\theta)^2 \, d\theta}{\int_0^\pi d\theta}.$$

Show that $$i_e = \frac{i_0}{\sqrt{2}}.$$

13. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

14. Find the area bounded by the curve $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$

15. Find the area between the cissoid $y^2 = \frac{x^3}{2a - x}$ and its asymptote $x = 2a$.

16. Find the area under the curve $y = \log x$ from $x = 1$ to $x = 10$.

17. Find the area of the loop of the curve $my^2 = (x - a)(x - b)^2$.

18. Find the length (a) of the curve $y = e^x$ from $x = 0$ to $x = 2$; (b) of the curve $y = \log x$ from $x = 1$ to $x = 6$; (c) of the curve $9y^2 = x(x-3)^2$ from $x = 0$ to $x = 3$.

19. Find the length of the spiral $\rho = a\theta$ from the origin to the point $(\pi a, \pi)$.

20. Find the length of an arc of the cissoid $\rho = 2a\dfrac{\sin^2\theta}{\cos\theta}$ from $\theta = 0$ to $\theta = \frac{1}{4}\pi$.

21. Find the length of the curve $\rho = a\sin^3\dfrac{\theta}{3}$.

22. A circle of radius $\dfrac{a}{2}$ rolls on a circle of radius a, and a point on the circumference of the rolling circle traces an epicycloid whose polar equation is

$$4(\rho^2 - a^2)^3 = 27a^4\rho^2\sin^2\theta.$$

Find the whole length of the curve thus traced.

23. Find the volume generated by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about the X-axis.

24. Find the volume generated by revolving one arch of the cycloid

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

about the axis OX.

25. Find the volume generated if the cycloid is revolved about OY.

26. Find the volume generated by revolving one arch of the curve $y = \cos x$ about the axis OX.

27. Find the volume generated by revolving the cardioid $\rho = 2a(1 - \cos\theta)$ about the axis OX.

28. A conoid is generated by a straight line which moves in such a manner as to intersect a given straight line and a given curve, at the same time remaining parallel to a given plane. Find the volume of a right conoid having an altitude h and a circular base of radius a.

29. Two cylinders, with the same altitude h, have a common upper base of radius a, and the lower bases are tangent to each other. Find the volume common to the two cylinders.

30. Find the surface generated by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about its major axis; also about its minor axis.

31. Find the surface generated by revolving the cardioid $\rho = 2a(1 - \cos\theta)$ about the initial line.

32. Find the mean length of the ordinates of a semicircle of radius a, if the ordinates are taken at equidistant intervals on the diameter.

33. A line of length a is divided into two segments. Prove that the mean area of the rectangle formed on the segments is $\frac{1}{6}a^2$.

34. Find the mean distance of the points on the circumference of a circle of radius a from a fixed point on the circumference.

35. Zeuner's equation for superheated steam is

$$pv = BT + Cp^n.$$

For an isothermal expansion the temperature T is constant. Derive an expression for the work done during an isothermal expansion from v_1 to v_2.

Suggestion: Use the equation

$$\text{Work} = \int p\,dv = pv - \int v\,dp.$$

CHAPTER XII

FUNCTIONS OF TWO OR MORE VARIABLES

101. Definition of a function of several variables. Heretofore we have discussed functions of a single independent variable. A function, however, may depend upon two or more variables having no mutual relation, that is, independent of one another. Thus the volume of a gas depends upon the temperature and also upon the pressure to which it is subjected, and the pressure and temperature may vary independently. In general, we may say:

z is a function of the independent variables $x, y, \ldots$ when for each set of values of these variables there is determined a definite value or values of z.

In what follows, we shall have occasion to deal, for the most part, with functions of two independent variables. We shall denote such functions symbolically by

$f(x, y)$, $F(x, y)$, $\phi(x, y)$, etc.

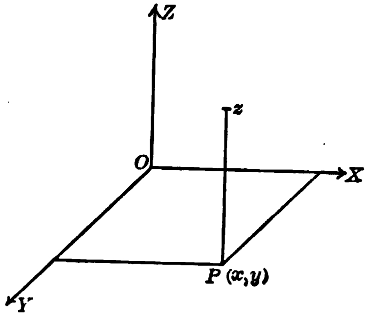

Fig. 80.

In order to interpret geometrically the relation $z = f(x, y)$, we must make use of space coördinates. To each position of the point $P(x, y)$ (Fig. 80) of the XY- plane, there corresponds a definite ordinate z. As P takes all positions in the XY- plane, the extremity of this moving z-ordinate generates a surface whose equation is

$$z = f(x, y).$$

The essential thing to be noted is that to every set of values $(x\ y)$ which enter into the discussion, there is a corresponding point on this surface.

The motion of a point on this surface depends upon the manner in which x and y vary. One of these variables may remain constant while the other is allowed to vary, or they may change simultaneously. In the first case, the extremity of the ordinate describes a plane curve lying in a plane parallel to the YZ- or XZ- plane; and in the second case the ordinate may describe a space curve. So long as the point on the surface moves along a curve lying in a plane parallel to either coördinate plane, that is, so long as we have a function of a single variable, there is no choice of path between any two points of that curve. When, however, the point is free to move upon the surface in any manner whatever between two fixed points, the motion is no longer restricted to a particular path. There is a closer restriction upon the motion of the point in one case than in the other.

102. Continuity of functions of two variables. The different ways in which a function of two variables may vary, as discussed in the previous article, introduces a distinction as to the kind of continuity the function may have. So long as one of the two variables remains fixed while the other is allowed to change, the only continuity involved is that of a single variable. Consequently no new principles are involved. Suppose, for example, that we consider the function $f(x, y)$ in the neighborhood of the point (x_0, y_0). If y remains constantly equal to y_0, we say that $f(x, y)$ is continuous with respect to x alone at this point, if

$$\underset{x \doteq x_0}{L} f(x, y_0) = f(x_0, y_0). \tag{1}$$

Similarly, if x remains constant, we say that the function is continuous with respect to y alone, provided

$$\underset{y \doteq y_0}{L} f(x_0, y) = f(x_0, y_0). \tag{2}$$

Another kind of continuity is introduced when we permit x and y to vary simultaneously. To distinguish this from that already discussed, we shall call it the continuity of the function with respect to both variables together. In general, whenever we speak of the continuity of a function of two variables, the continuity with respect to both variables is to be understood unless otherwise stated. The condition that a function $f(x, y)$ shall be continuous with respect to both x and y at any point (x_0, y_0) is that we have

$$\underset{\substack{x \doteq x_0 \\ y \doteq y_0}}{L} f(x, y) = f(x_0, y_0). \tag{3}$$

The essential characteristics of this condition are that the function must approach the same limiting value as the variables approach the limiting point in any manner whatever, and, moreover, that this limiting value shall be the value of the function at that point. When a simultaneous limit exists, it follows that the single limits $\underset{x \doteq x_0}{L} f(x, y_0)$ and $\underset{y \doteq y_0}{L} f(x_0, y)$ also exist and give the same limiting value. Hence, if a function is continuous with respect to both variables taken together, it is also continuous with respect to each taken separately. The converse of the statement does not hold; that is, there are functions which are continuous with respect to each variable separately, yet are not continuous in both together. The following examples will illustrate this distinction.

Ex. 1. Discuss the continuity of the function $f(x, y) = \frac{xy}{x^2 + y^2}$, where $f(0, 0)$ is defined as 0.

We have for the point (0, 0),

$$\underset{x \doteq 0}{L} f(x, 0) = \underset{x \doteq 0}{L} \frac{x \cdot 0}{x^2 + 0^2} = 0 = f(0, 0),$$

$$\underset{y \doteq 0}{L} f(0, y) = \underset{y \doteq 0}{L} \frac{0 \cdot y}{0 + y^2} = 0 = f(0, 0),$$

If the point (x, y) approaches the origin along the line $x = 0$, or along the line $y = 0$, the limiting value of the function is therefore 0, the value of the function. If, however, (x, y) approaches the origin along the line $x = y$, the limiting value is

$$\underset{x \doteq 0}{L} \frac{x^2}{2x^2} = \frac{1}{2},$$

which is different from zero. Hence at the origin the given function is continuous with respect to each variable separately but not with respect to both variables taken together. At all other points it is continuous with respect to both variables taken together and hence with respect to each separately.

Ex. 2. Given $f(x, y) = x^2 + y^2$. Here we have for the point $x = y = 0$,

$$\underset{\substack{x \doteq 0 \\ y \doteq 0}}{L} (x^2 + y^2) = 0 = f(0, 0).$$

Hence, at the origin, as at all other points, the function is continuous with respect to both variables, and, consequently, with respect to each separately.

EXERCISES.

1. Investigate the continuity of the function $f(x, y) = \frac{x - y}{x + y}$, $f(0, 0)$ being defined as 0. Why is it necessary to define $f(x, y)$ for $x = y = 0$? How about the continuity when we define $f(0, 0) = 4$?

2. Investigate the continuity of the function $f(x, y) = \dfrac{2xy^2}{x^2 + y^4}$, where $f(0, 0)$ is defined as 0.

3. Investigate the continuity of the function $x^2 - 2y^2$ at the origin.

103. Partial derivatives. In the previous article it was shown that a function of two variables may vary either by permitting one of the variables to remain constant while the other changes, or by allowing both to vary simultaneously. The increment of the function $f(x, y)$ due to a change in x alone is

$$f(x + \Delta x, y) - f(x, y).$$

Let us consider the ratio of this increment to the increment of the variable x. The limit of this ratio as $\Delta x \doteq 0$; viz.,

$$\underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

is called the *partial derivative* of $f(x, y)$ with respect to x. Similarly, the limit

$$\underset{\Delta y \doteq 0}{L} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

is called the partial derivative of $f(x, y)$ with respect to y. These derivatives are called partial derivatives because they measure only partially the variation of the function as compared with that of the variables. To distinguish these from the derivatives which have been thus far considered, we shall call the latter *total derivatives*. In a subsequent article we shall discuss methods of determining total derivatives of functions of several variables.

To distinguish symbolically the two classes of derivatives, we shall denote the partial derivatives by using the round ∂ instead of d. Thus the partial derivatives of $z = f(x, y)$ with respect to x and y will be written

$$\frac{\partial z}{\partial x}, \ \frac{\partial z}{\partial y},$$

respectively. They are frequently represented also by the equivalent symbols

$$f'_x(x, y), f'_y(x, y).$$

Partial derivatives are, in general, functions of x and y, and may likewise have partial derivatives with respect to either variable. Thus we may have

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right), \ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right), \ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

These higher partial derivatives are represented symbolically by

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x\, \partial y}, \quad \frac{\partial^2 z}{\partial y\, \partial x},$$

or by

$$f''_x(x, y), f''_y(x, y), \; f''_{xy}(x, y), f''_{yx}(x, y),$$

respectively.

We may extend these considerations to partial derivatives of still higher order and to functions of more than two variables. The notation employed for the partial derivatives of higher order indicates the number of differentiations and the order in which they are made. Thus, $\frac{\partial^3 u}{\partial x\, \partial y^2}$ indicates three differentiations, the first two with respect to y, the third with respect to x; $\frac{\partial^5 u}{\partial z\, \partial x^2\, \partial y^2}$ indicates five differentiations, the first and second with respect to y, the third and fourth with respect to x, and the fifth with respect to z.

It has been shown that the function $z = f(x, y)$ represents a surface, while $z = f(x, b)$ gives a plane curve cut from this surface by the plane $y = b$. For the derivative $\frac{\partial z}{\partial x}$ we have the same geometric interpretation in this plane as was given earlier for $\frac{dz}{dx}$, namely, the slope of the tangent to the curve in question.

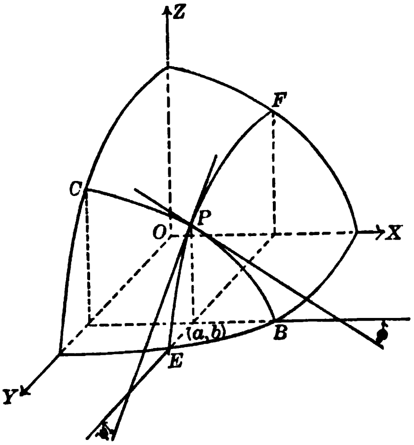

Fig. 81.

Let $z = f(x, y)$ be the surface shown in Fig. 81. Consider any point on this surface, as P, at the intersection of the curves BPC and EPF, cut from the surface by the planes $y = b$ and $x = a$, respectively. Then the slope of the curve BPC is given by the partial derivative $\frac{\partial z}{\partial x}$, and that of the curve EPF by the partial derivative $\frac{\partial z}{\partial y}$. That is,

$$\tan \phi = \frac{\partial z}{\partial x}, \quad \tan \psi = \frac{\partial z}{\partial y}.$$

The values of tan ϕ and tan ψ for some definite point P on the surface are obtained by substituting in the expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, respectively, the corresponding values of x and y. Thus if (a, b) is the projection of P on the XY-plane, we substitute a for x and b for y.

The process of finding a partial derivative is in all respects the same as that employed in finding an ordinary derivative of a function of a single variable. The following examples illustrate this statement.

Ex. 1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, when $z = y^2 \sin x$.

Treating y as a constant and differentiating with respect to x, we have

$$\frac{\partial z}{\partial x} = y^2 \cos x.$$

Likewise, if we consider x constant and differentiate with respect to y, we get

$$\frac{\partial z}{\partial y} = 2\,y \sin x.$$

Ex. 2. Find successive partial derivatives of the function $xe^x \log y$.

We have $$z = xe^x \log y.$$

$$\frac{\partial z}{\partial x} = (1 + x)\,e^x \log y; \qquad \frac{\partial z}{\partial y} = \frac{xe^x}{y}.$$

$$\frac{\partial^2 z}{\partial x^2} = (2 + x)\,e^x \log y; \qquad \frac{\partial^2 z}{\partial y^2} = -\frac{xe^x}{y^2}.$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{(1 + x)e^x}{y}; \qquad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{(1 + x)e^x}{y}.$$

Ex. 3. The surface $z = \frac{x^2}{8} + \frac{y^2}{12}$ is cut by planes $x = 4$, $y = 3$. Find the slopes of the curves cut from the surface by these planes at the point of intersection of the curves.

We have here

$$\frac{\partial z}{\partial x} = \frac{x}{4}, \qquad \frac{\partial z}{\partial y} = \frac{y}{6}.$$

At the specified point, therefore,

$$\tan\phi = \left.\frac{x}{4}\right]_{x=4} = 1; \quad \tan\psi = \left.\frac{y}{6}\right]_{y=3} = \frac{1}{2}.$$

EXERCISES.

1. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following functions:

(a) $z = x^2y^5$.
(b) $z = \sin x \cos y$.
(c) $z = ye^x + xe^y$.
(d) $z = 3x^5 - 4x^3 + x - 17$.
(e) $z = y^x$.
(f) $z = \tan \frac{y}{x}$.
(g) $z = y \sin x + x \sin y$.
(h) $z = \cot (x^2 - y^2)$.

2. Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ in the following functions of three variables.

(a) $u = 5xy^2z^{-\frac{1}{3}}$.
(b) $u = e^x \log yz$.
(c) $u = \sin x \cos y + \sin y \cos z + \sin z \cos x$.
(d) $u = \log \frac{x+y}{z}$.
(e) $u = \sin (yz + zx + xy)$.

3. The volume of a cone may be expressed as a function of the radius of the base and of the altitude. Denoting the volume by V, the base radius by r, and the altitude by h, express V in terms of r and h, find the partial derivatives $\frac{\partial V}{\partial r}$, $\frac{\partial V}{\partial h}$, and give interpretations of these derivatives.

4. Express the area A of a triangle as a function of its base x and altitude y. Find the rate at which the area changes: (a) when the altitude remains unchanged and the base varies; (b) when the base is constant and the altitude varies.

5. Interpret geometrically the equations

$$\frac{\partial (xy)}{\partial x} = y, \qquad \frac{\partial (xy)}{\partial y} = x.$$

6. Form $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \, \partial y}$, $\frac{\partial^2 u}{\partial y \, \partial x}$, and $\frac{\partial^2 u}{\partial y^2}$ for the following functions.

(a) $u = \tan xy^3$.
(b) $u = x \log y$.
(c) $u = x^my^n$.
(d) $u = x^3 + xy - y^3$.
(e) $u = e^{x+y}$.
(f) $u = x(1 - y^3)$.

7. The formula $H = ks^3D^{\frac{2}{3}}$ is used to determine the horsepower required to drive a steamship in terms of the speed s and displacement D. What is the interpretation of the derivative $\frac{\partial H}{\partial s}$? of $\frac{\partial H}{\partial D}$? Derive expressions for these derivatives.

104. Interchange of order of differentiation. Among the partial derivatives enumerated in Art. 103 were the second derivatives $\frac{\partial^2 f}{\partial x \, \partial y}$, $\frac{\partial^2 f}{\partial y \, \partial x}$. In most cases that arise in the applications of the calculus to physical problems it is a matter of indifference in what order the differentiation is performed; that is, in most cases these two partial derivatives give the same result. This, however, is not always true, and it is desirable to have some criterion for the equality of these derivatives.

Let us consider the limits involved. We have by definition,

$$\frac{\partial f}{\partial x} = \underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}, \tag{1}$$

$$\frac{\partial f}{\partial y} = \underset{\Delta y \doteq 0}{L} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}. \tag{2}$$

Consequently we have

$$\begin{aligned}\frac{\partial^2 f}{\partial y\, \partial x} &= \underset{\Delta y \doteq 0}{L}\ \underset{\Delta x \doteq 0}{L} \frac{\dfrac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} - \dfrac{f(x+\Delta x, y) - f(x, y)}{\Delta x}}{\Delta y} \\ &= \underset{\Delta y \doteq 0}{L}\ \underset{\Delta x \doteq 0}{L} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x, y)}{\Delta y\ \Delta x};\end{aligned} \tag{3}$$

and similarly,

$$\frac{\partial^2 f}{\partial x\, \partial y} = \underset{\Delta x \doteq 0}{L}\ \underset{\Delta y \doteq 0}{L} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x, y)}{\Delta y\ \Delta x}. \tag{4}$$

It appears that the only difference between the two derivatives is the order in which the increments Δx and Δy are allowed to approach zero. As has been pointed out, this order is not always a matter of indifference. The following theorem gives a helpful criterion in such cases.

THEOREM. *The order of differentiation can be interchanged if in the neighborhood of the point in question* $\dfrac{\partial f}{\partial x}$ *and* $\dfrac{\partial^2 f}{\partial y\, \partial x}$ $\left(\text{or } \dfrac{\partial f}{\partial y} \text{ and } \dfrac{\partial^2 f}{\partial x\, \partial y}\right)$ *are continuous functions of both variables* x *and* y *together.*

If we at first consider y and Δy as constants, we may put

$$F(x) = f(x, y + \Delta y) - f(x, y), \tag{5}$$

and the numerator of (4) becomes

$$F(x + \Delta x) - F(x). \tag{6}$$

We have from the law of the mean,

$$F(x + \Delta x) - F(x) = \Delta x\, F'_x(x + \theta_1 \Delta x),\quad 0 < \theta_1 < 1. \tag{7}$$

Expressed in terms of $f(x, y)$, the right-hand member of this equation becomes

$$\Delta x\,[f'_x(x + \theta_1 \Delta x, y + \Delta y) - f'_x(x + \theta_1 \Delta x, y]. \tag{8}$$

Regarding now $x + \theta_1 \Delta x$ as a constant determined by (7), and y as a variable, we have from the application of the law of the mean to the function in the brackets,

$$\begin{aligned} &f'_x(x + \theta_1 \Delta x, y + \Delta y) - f'_x(x + \theta_1 \Delta x, y) \\ &= f''_{yx}(x + \theta_1\, \Delta x, y + \theta_2 \Delta y), \qquad 0 < \theta_2 < 1. \end{aligned} \tag{9}$$

From (7), (8), (9), we may now write the limit in the right-hand member of (4) as follows:

$$\begin{aligned} &\underset{\Delta x \to 0}{L}\ \underset{\Delta y \to 0}{L} \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) - f(x + \Delta x, y) + f(x, y)}{\Delta x\ \Delta y} \\ &\quad = \underset{\Delta x \to 0}{L}\ \underset{\Delta y \to 0}{L}\ f''_{yx}(x + \theta_1 \Delta x, y + \theta_2 \Delta y). \end{aligned} \tag{10}$$

By hypothesis, $f''_{yx}(x, y)$ is a continuous function in both x and y together. Consequently, the right-hand member of the above equation approaches $f''_{yx}(x, y)$ in whatever order Δx and Δy are allowed to approach zero. The left-hand member of (10) approaches by hypothesis the limit $f''_{xy}(x, y)$. Upon passing to the limit, we have, therefore,

$$f''_{xy}(x, y) = f''_{yx}(x, y), \tag{11}$$

which establishes the theorem.

EXERCISES

1. In each of the following functions show that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$.

(a) $u = x^3y^2 - 4\,xy^4$.

(b) $u = \cos\,(x + y)$.

(c) $u = e^x \sin y$.

(d) $u = y^x$.

(e) $u = \log\,(x + y^2)$.

(f) $u = \cos xy^2$.

2. For each of the following functions show that $\dfrac{\partial^3 u}{\partial x\, \partial y^2} = \dfrac{\partial^3 u}{\partial y^2\, \partial x}$.

(a) $u = x^2\,(x - y^4)$.

(b) $u = xy \cos\,(x + y)$.

(c) $u = y \log\,(1 + xy)$.

(d) $u = \sin^2 x \cos y$.

105. Multiple integrals. In Chapter VIII we considered successive differentiation and successive integration of functions of a single variable. In the present chapter we have discussed the successive partial differentiation of functions of two or more variables. Let us now consider the inverse problem of successive integration of functions of several variables.

Suppose, for example, that we have given a function $f(x, y, z)$ of three independent variables. We may write

$$\int f(x, y, z)\, dz = F(x, y, z), \tag{1}$$

$$\int F(x, y, z)\, dy = F_1(x, y, z), \tag{2}$$

$$\int F_1(x, y, z)\, dx = F_2(x, y, z), \tag{3}$$

where in (1) the integration is taken with respect to z, that is, as if y and x were constants. Likewise in (2) it is taken with respect to y, and in (3) with respect to x. Substituting in (3) the value of $F_1(x, y, z)$ from (2), we have

$$F_2(x, y, z) = \int \left[\int F(x, y, z)\, dy \right] dx, \tag{4}$$

and by the use of (1) this becomes

$$F_2(x, y, z) = \int \left\{ \int \left[\int f(x, y, z)\, dz \right] dy \right\} dx. \tag{5}$$

The expression (5) may be written more compactly as follows:

$$F_2(x, y, z) = \int \int \int f(x, y, z)\, dz\, dy\, dx. \tag{6}$$

It is to be understood that the integral is to be first taken with respect to z, then y, and finally x.*

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* Books on the calculus differ in the manner of indicating the order in which the integrals are to be taken. Some authors write the differential last which is to be taken in the first integration, etc., that is, the order of the differentials in equation (6) is exactly reversed. The above notation is adopted in this book because it shows best the manner in which the integration has arisen. In other works, the context will usually indicate to the reader the notation employed.

An expression like the second member of (6), which indicates the result of several successive integrations, is called a *multiple integral.* If two integrations are involved, the integral is called a *double integral;* if three, a *triple integral,* etc.

The evaluations of an indefinite multiple integral differs from that of an indefinite single integral in one essential feature; namely, the form of the constant of integration. An example will illustrate this point.

Given
$$\int\int e^{2x}y^2 dy\, dx;$$

we are to find a function u of x and y such that

$$\frac{\partial^2 u}{\partial x\, \partial y} = e^{2x}y^2.$$

Integrating first with respect to y regarding x as constant, we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{3}e^{2x}y^3 + \text{constant of integration.}$$

Now since x was considered as constant during the integration, the constant of integration may depend upon x. To make this more clear, we may observe that differentiating either

$$\frac{1}{3}e^{2x}y^3 + C,$$

or
$$\frac{1}{3}e^{2x}y^3 + \phi(x),$$

with respect to y gives the same result, $e^{2x}y^2$. Hence we assume the more general case and write

$$\frac{\partial u}{\partial x} = \frac{1}{3}e^{2x}y^3 + \phi(x),$$

where $\phi(x)$ is an arbitrary function of x. As a special case, $\phi(x)$ may of course be a constant C.

Integrating this result, keeping y constant, we obtain

$$u = \frac{1}{6}e^{2x}y^3 + \int\phi(x)\, dx + \psi(y).$$

Here again, since y was considered constant during the integration, the integration constant must be taken as an arbitrary function of y.

By assigning limits of integration to each variable, we arrive at the notion of a definite multiple integral. Thus the integral

$$\int_0^1 \int_a^b e^{2x} y^2 \, dy \, dx,$$

indicates that $e^{2x}y^2$ is to be integrated first between the limits a and b with respect to y, and the result thus found is to be integrated with respect to x between the limits 0 and 1.

Since x is considered as constant in the integration with respect to y, the y-limits, a and b, may be functions of x. Similarly in a definite triple integral, taken first with respect to z, then y, and finally x, the z-limits may be functions of both y and x, and the y-limits functions of x.

Ex. Evaluate $\displaystyle\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{a \cos \theta} r^2 \sin \theta \, dr \, d\theta \, d\phi$

The first integration with respect to r gives

$$\frac{1}{3} a^3 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos^3 \theta \sin \theta \, d\theta \, d\phi.$$

Integrating with respect to θ, we obtain

$$\frac{1}{12} a^3 \int_0^{2\pi} d\phi = \frac{\pi}{6} a^3.$$

It should be observed that the upper limit of the r-integral is a function of θ.

EXERCISES.

Evaluate the following integrals:

1. $\displaystyle\int\int x^2 y \, dy \, dx.$

2. $\displaystyle\int\int e^x \sin y \, dy \, dx.$

3. $\displaystyle\int_2^5 \int_0^3 \int_1^4 x^3 y z^2 \, dz \, dy \, dx.$

4. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\sin \theta} r \, dr \, d\theta.$

5. $\displaystyle\int_0^{\pi} \int_0^{a(1-\sin \theta)} r^2 \sin \theta \, dr \, d\theta.$

6. $\displaystyle\int_0^{\frac{\pi}{2}} \int_{a(1-\cos \theta)}^{a} \rho \, d\rho \, d\theta.$

7. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{2a \cos \theta} \rho^3 \cos^2 \theta \, d\rho \, d\theta.$

8. $\displaystyle\int_0^{a} \int_0^{cz} \int_0^{\sqrt{c^2z^2-y^2}} (z^2 + y^2) \, dx \, dy \, dz.$

9. $\displaystyle\int_0^{b} \int_{h_1}^{h_2} \sqrt{2gy} \, dy \, dx.$

10. $\displaystyle\int_0^{2a} \int_0^{\arccos \frac{\rho}{2a}} \rho \, d\theta \, d\rho.$

11. $\displaystyle\int_0^{a} \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 y \, dy \, dx.$

12. $\displaystyle\int_0^{a} \int_0^{2\sqrt{ax}} x^n dy \, dx.$

106. Total derivatives. Let $z = f(x, y)$ be a continuous function of the two variables, having continuous derivatives. Moreover, let both x and y be continuous functions of a common variable t, having the continuous derivatives $\frac{dx}{dt}, \frac{dy}{dt}$. We shall now study the relation of the variation of z to the variations of x and y; in other words, we shall find an expression for $\frac{dz}{dt}$ in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

For the increment of z we have

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y),$$

which may be written in the following form:

$$\Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]. \quad (1)$$

By an application of the law of the mean, we have

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = f_x'(x + \theta_1 \Delta x, y + \Delta y)\,\Delta x, \quad (2)$$

$$f(x, y + \Delta y) - f(x, y) = f_y'(x, y + \theta_2 \Delta y)\,\Delta y, \quad (3)$$

where θ_1, θ_2 lie between 0 and 1. Substituting these values in the second member of (1), we have after dividing by Δt,

$$\frac{\Delta z}{\Delta t} = f_x'(x + \theta_1 \Delta x, y + \Delta y)\frac{\Delta x}{\Delta t} + f_y'(x, y + \theta_2 \Delta y)\frac{\Delta y}{\Delta t}. \quad (4)$$

In the limit, we have

$$\underset{\Delta t \doteq 0}{L} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}, \quad \underset{\Delta t \doteq 0}{L} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}, \quad \underset{\Delta t \doteq 0}{L} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}. \quad (5)$$

Moreover, since Δx and Δy approach zero with Δt, and $f_x'(x, y)$ and $f_y'(x, y)$ are continuous functions of both variables together, we have

$$\underset{\Delta t \doteq 0}{L} f_x'(x + \theta_1 \Delta x, y + \Delta y) = f_x'(x, y), \quad (6)$$

$$\underset{\Delta t \doteq 0}{L} f_y'(x, y + \theta_2 \Delta y) = f_y'(x, y). \quad (7)$$

Writing $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for $f_x'(x, y)$, $f_y'(x, y)$ respectively, we have from (4)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \quad (a)$$

Because of the importance of this equation, the following geometrical illustration is added. Let $PRQS$, Fig. 82, be an element of the surface $z = f(x, y)$. Taking the coördinates of P as (x_1, y_1, z_1) and those of Q as (x_2, y_2, z_2), we have

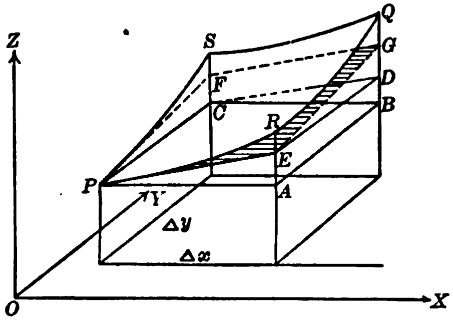

Fig. 82.

$$PA = x_2 - x_1 = \Delta x,$$
$$PC = y_2 - y_1 = \Delta y,$$
$$BQ = z_2 - z_1 = \Delta z.$$

Suppose a tangent plane $PEGF$ passed through P; then PE and PF are tangent at P to the curves PR and PS respectively. Suppose further a plane passed through the points CEP, cutting BQ in D. Then from the figure, we have

$$BD = AE, \qquad DG = CF,$$

and
$$BG = BD + DG = AE + CF.$$

But
$$AE = PA \tan EPA = \Delta x \frac{\partial z}{\partial x},$$

and
$$CF = PC \tan FPC = \Delta y \frac{\partial z}{\partial y};$$

hence
$$BG = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y,$$

or
$$\frac{BG}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t}.$$

The increments Δx and Δy approach zero with Δt, and the ratio $\frac{BG}{\Delta t}$ in the limit becomes equal to the ratio $\frac{BQ\,(=\Delta z)}{\Delta t}$.

We have then

$$\underset{\Delta t \doteq 0}{L} \frac{BG}{\Delta t} = \underset{\Delta t \doteq 0}{L} \frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x} \underset{\Delta t \doteq 0}{L} \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \underset{\Delta t \doteq 0}{L} \frac{\Delta y}{\Delta t};$$

whence
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

This result may be expressed in words as follows:

The total rate of change of a function of x and y is made up of two parts: one is the rate of change of x multiplied by the partial *x-derivative of the function, the other the rate of change of y multiplied by the partial y-derivative.*

The following examples will serve to illustrate this principle.

Ex. **1.** Suppose the equation of a gas to be $pv = 54\,T$, and let the volume and temperature at a given instant be $v_0 = 15$ and $T_0 = 640$. The corresponding pressure is

$$p_0 = \frac{54 \times 640}{15} = 2304.$$

In this state suppose the temperature to be rising at the rate of 0.5 degree per second and the volume to be increasing at the rate of 0.2 cubic foot per second. Required the rate at which the pressure is changing.

We have
$$p = 54\frac{T}{v},$$

whence
$$\frac{\partial p}{\partial T} = \frac{54}{v}, \quad \frac{\partial p}{\partial v} = -54\frac{T}{v^2}.$$

Hence in the given state,

$$\left.\frac{\partial p}{\partial T}\right]_{T=T_0} = \frac{54}{15} = 3.6,$$

and
$$\left.\frac{\partial p}{\partial v}\right]_{v=v_0} = -\frac{54 \times 640}{15^2} = -153.6.$$

Also taking the time t as the auxiliary variable,

$$\frac{dT}{dt} = 0.5, \text{ and } \frac{dv}{dt} = 0.2.$$

Then
$$\frac{dp}{dt} = \frac{\partial p}{\partial T}\frac{dT}{dt} + \frac{\partial p}{\partial v}\frac{dv}{dt} = 3.6 \times 0.5 - 153.6 \times 0.2 = -28.92.$$

That is, the pressure is decreasing at the rate of 28.92 pounds per square foot per second.

Ex. **2.** The equation
$$z = 2x^2 + 5y^2$$

represents an elliptic paraboloid. At the point $x = -3$, $y = 1$, we have

$$\left.\frac{\partial z}{\partial x}\right]_{x=-3} = 4x\Big]_{x=-3} = -12$$

$$\left.\frac{\partial z}{\partial y}\right]_{y=1} = 10y\Big]_{y=1} = 10.$$

Assume the rate of change of x to be 3 units per second, and that of y to be 2 units per second; that is, $\frac{dx}{dt} = 3$ and $\frac{dy}{dt} = 2$. Then the rate of change of z is

$$\frac{dz}{dt} = -12\frac{dx}{dt} + 10\frac{dy}{dt} = -16 \text{ units per second.}$$

In the preceding discussion x and y were taken as independent; that is, each was defined by an arbitrary function, and the function of t which defined x had no definite relation to the function of the same variable which defined y. Under these circumstances, (x, y) is *any* point in the XY- plane, and as x and y vary, the moving point is not restricted to a particular curve. The corresponding values which z assumes include all positions on the surface $z = f(x, y)$.

Let us now suppose that there is a relation existing between x and y. This relation may be expressed explicitly by the equation

$$y = F(x),$$

or implicitly by the relation

$$\phi(x, y) = 0.$$

These relations restrict the moving point (x, y) to a particular curve in the XY- plane and consequently the values of z to a particular curve in space, that is, to the intersection of the cylinder $\phi(x, y) = 0$ and the surface $z = f(x, y)$. Whenever such a relation as the above exists between x and y, we may choose x itself as the variable t, and formula (a) reduces to the following:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\,\frac{dy}{dx}. \qquad (b)$$

In this formula it is to be observed that $\frac{dz}{dx}$ and $\frac{\partial z}{\partial x}$ have very different meanings. The partial derivative $\frac{\partial z}{\partial x}$ is formed on the supposition that x alone varies, that is, that y is constant. On the other hand, $\frac{dz}{dx}$ expresses the variation of z due to a change in both x and y; for this reason it is called a *total derivative*. Likewise in (a), $\frac{dz}{dt}$ is the total derivative of z with respect to t. Moreover, the value of $\frac{\partial z}{\partial x}$ is definitely determined by the value of the coördinates of the point in question; in other words, it is a function of the coördinates only. This follows from the fact that the variation can take place in one direction only. Functions like this, which depend for their values solely upon the coördinates of the point, are called *point functions*. The value of $\frac{dz}{dx}$ depends not only upon the coördinates of the point but also upon the direction in which that point is approached. This derivative is therefore not a point function.

Formulas (*a*) and (*b*) may be extended to functions of any number of variables. Thus, if

$$u = f(x, y, z),$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}; \qquad (c)$$

and if further $y = \phi(x)$, $z = \psi(x)$,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}. \qquad (d)$$

The proof is left as an exercise for the student.

Formulas (*b*) and (*d*) are chiefly useful in the differentiation of somewhat complicated functions of a single variable. The following example shows such an application.

Ex. 3. Find $\frac{du}{dx}$, where $u = xe^{\sqrt{a^2-x^2}}\sin^3 x$.

Let $y = \sqrt{a^2 - x^2}$ and $z = \sin^3 x$; then $u = xe^y z$,

and
$$\frac{\partial u}{\partial x} = e^y z, \quad \frac{\partial u}{\partial y} = xe^y z, \quad \frac{\partial u}{\partial z} = xe^y.$$

Also
$$\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}}, \quad \frac{dz}{dx} = 3\sin^2 x\cos x.$$

Substituting these expressions for the derivatives in (*d*), we get

$$\begin{aligned}\frac{du}{dx} &= e^y z - \frac{x^2 e^y z}{\sqrt{a^2 - x^2}} + 3\,xe^y \sin^2 x\cos x \\ &= e^{\sqrt{a^2-x^2}}\sin^3 x\left[1 - \frac{x^2}{\sqrt{a^2 - x^2}} + 3\,\cot x\right].\end{aligned}$$

EXERCISES.

1. Find $\frac{du}{dx}$ in each of the following:

   (a) $u = x^2 + y^2$, and $y = e^{\sin x}$.

   (b) $u = \arctan\frac{y}{x}$, and $y = e^x$.

   (c) $u = \log(x + y)$, and $y = \sqrt{x^2 + a^2}$.

   (d) $u = x^2 e^{\frac{1}{x}}\cos x$.

   (e) $u = e^{ax}(y - z)$, and $y = a\sin x$, $z = \cos x$.

2. A triangle has a base of 10 units and an altitude of 6 units. The base is made to increase at the rate of 2 units, and the altitude to decrease at the rate of ½ unit. At what rate does the area change?

3. A point lying on the ellipsoid $\frac{x^2}{36} + \frac{y^2}{25} + \frac{z^2}{49} = 1$ in the position $x = 3$, $y = -4$, moves so that x increases at the rate of two units per second, while y decreases at the rate of three units per second. Find the rate of change of z.

4. With the same data as in Ex. 1, Art. 106, suppose the pressure of the gas to be increasing at the rate of 40 pounds per square foot per second, while the temperature is falling at the rate of 1 degree per second. Find the rate of change of the volume.

5. A gas has the equation $pv = RT$, and expands following the law $pv^n = C$, where n and C are constants. Derive expressions for the total derivatives $\frac{dT}{dv}$ and $\frac{dT}{dp}$ under these conditions.

6. Derive the same results by eliminating p and v successively between the equation of the gas and the equation of the expansion and differentiating the resulting expressions.

107. Total differentials. In formula (a), $\frac{dz}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$ are the quotients of two differentials; hence we may multiply through by the differential dt. The resulting formula

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \qquad (e)$$

expresses the differential of z in terms of the differentials of x and y. This formula may be extended to any number of variables. Thus if

$$u = f(x_1, x_2, x_3, \ldots, x_n),$$

$$du = \frac{\partial u}{\partial x_1}\,dx_1 + \frac{\partial u}{\partial x_2}\,dx_2 + \frac{\partial u}{\partial x_3}\,dx_3 + \cdots + \frac{\partial u}{\partial x_n}\,dx_n.$$

The differential dz in the formula (e) is called the *total differential* of z. The terms $\frac{\partial z}{\partial x}\,dx$ and $\frac{\partial z}{\partial y}\,dy$ are called respectively the *partial x-differential* and *partial y-differential* of z. These latter are frequently denoted by d_xz and d_yz, and (e) is then written in the form

$$dz = d_xz + d_yz. \qquad (e')$$

Equation (e) therefore expresses symbolically the principle that *the total differential of a function of several variables is equal to the sum of the partial differentials.*

Ex. Let $u = e^x y^2 \sin z$. Then

$$\frac{\partial u}{\partial x} = e^x y^2 \sin z, \quad \frac{\partial u}{\partial y} = 2\, e^x y \sin z, \quad \frac{\partial u}{\partial z} = e^x y^2 \cos z.$$

Therefore $$du = e^x y^2 \sin z\, dx + 2\, e^x y \sin z\, dy + e^x y^2 \cos z\, dz.$$

EXERCISES.

1. Differentiate each of the following functions by means of formula (*e*) and verify the result by direct differentiation.

(a) $z = e^x y^2$.
(b) $z = y^x$.
(c) $z = \sin x \cos y$.
(d) $z = \frac{x^3}{y^3} \cdot$
(e) $z = a^x e^y$.
(f) $z = \text{arc tan } \frac{x}{y} \cdot$
(g) $z = x^3 + y^3$.
(h) $z = \text{arc cos } \frac{y}{x} + \text{arc tan } \frac{x}{y} \cdot$

2. Differentiate each of the following functions.

(a) $u = x^3 + y^3 + 2\, xyz$.
(b) $u = z^{xy}$.
(c) $u = \tan \theta^2 \tan \phi^2 \tan \psi^2$.
(d) $u = \sin x \cos y \cos^2 z$.
(e) $u = \text{arc tan } \frac{xy}{z} \cdot$
(f) $u = \frac{x^2 z}{a^2 - y} \cdot$

3. From the perfect gas equation $pv = RT$, find the change in p produced by simultaneous changes of v and T.

4. The formula for the coefficient of diffusion of a gas is $k = CpT^m$, where C is a constant, p the pressure, and T the absolute temperature. Find the change in the coefficient due to simultaneous changes in pressure and temperature.

5. Taking the formula $H = ks^3 D^{\frac{2}{3}}$ for the horsepower of a steamship, derive an expression for the increase in horsepower due to an increase Δs in the speed, and an increase ΔD in the displacement.

108. Differentiation of implicit functions. Suppose that y is defined implicitly as a function of x by means of the relation $f(x, y) = 0$. In Art. 49, we discussed a method for obtaining the derivative $\frac{du}{dx}$ when the particular analytic expression for the given function is known. By applications of the principles deduced in Art. 107, we can now establish a general formula for writing out at once the required derivative.

Since the function $f(x, y)$ has by definition the constant value zero for all values of x and y, the total differential must also be zero. Hence we have

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0. \tag{1}$$

Transposing the first term to the second member of the equation and dividing both members by dx, we have

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}, \tag{f}$$

which may be used as a formula for writing out at once the required result.

Ex. Given $f(x, y) = x^2y - xy^3 = 0$.

We have $$\frac{\partial f}{\partial x} = 2\,xy - y^3, \quad \frac{\partial f}{\partial y} = x^2 - 3\,xy^2;$$

therefore, $$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = -\frac{2\,xy - y^3}{x^2 - 3\,xy^2}.$$

EXERCISES.

For each of the following functions find the first derivative by the method of this article.

1. $y^3 - 3\,x^2y + xy^2 = 0$.
2. $x^3 \sin y - y^3 \cos x = 0$.
3. $xy^n = C$.
4. $p\,(v - b)^n = C$, find $\dfrac{dp}{dv}$.
5. $\rho^3 - \rho^2 \cos\theta = C$, find $\dfrac{d\rho}{d\theta}$.
6. $b^2x^2 + a^2y^2 = a^2b^2$.

109. Approximation formulas. Formula (e) gives an exact relation between certain differentials. By reference to Fig. 82, we may derive an approximate relation of the same form connecting the corresponding increments. From Art. 106, we have

$$BG = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y. \tag{1}$$

As in Art. 46, we may take the differentials dx and dy of the independent variables as numerically equal to the increments Δx and Δy. Then (1) becomes

$$BG = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy. \tag{2}$$

Comparing (2) with equation (e), it appears that BG represents the total differential dz; while the increment Δz is represented by the segment BQ.

In general, therefore, dz and Δz are different. However, the smaller the increments Δx and Δy are taken, the smaller the relative difference, that is, the nearer the ratio $\frac{dz}{\Delta z}$ approaches 1. Hence, if in (1) we write Δz for BG, the resulting equation

$$\Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y \tag{3}$$

gives a relation which is approximate but which becomes more nearly exact, the smaller Δx and Δy are taken. This equation is useful in calculating an approximate value of Δz corresponding to arbitrary changes Δx; Δy. It may also be used in calculating the approximate effect of small errors. Thus, if z is a function of x and y, the error in z due to errors Δx and Δy in the observed values of x and y, is given approximately by (3). In general, if $u = f(x_1, x_2, x_3, \ldots, x_n)$, the error in u due to errors in the value of $x_1, x_2, x_3, \ldots, x_n$, is given approximately by

$$\Delta u = \frac{\partial u}{\partial x_1} \Delta x_1 + \frac{\partial u}{\partial x_2} \Delta x_2 + \ldots + \frac{\partial u}{\partial x_n} \Delta x_n. \tag{4}$$

Ex. 1. If $z = x^2 - xy$, find approximately the increment of z, corresponding to $\Delta x = .02$, $\Delta y = .01$, when $x = 8$, $y = 5$.

From (3), we have approximately

$$\Delta z = (2x - y)\Delta x - x\Delta y = .22 - .08 = .14.$$

The actual change of z is

$$(8.02^2 - 8.02 \times 5.01) - (8^2 - 8 \times 5) = .1402.$$

Ex. 2. Let W denote the weight of a body in air, and w its weight in water; then the formula

$$S = \frac{W}{W - w}$$

gives the specific gravity of the body.

Since $\frac{\partial S}{\partial W} = -\frac{w}{(W-w)^2}$ and $\frac{\partial S}{\partial w} = \frac{W}{(W-w)^2}$, the error ΔS in the calculated specific gravity due to errors ΔW and Δw in the weights is

$$\Delta S = -\frac{w}{(W-w)^2}\Delta W + \frac{W}{(W-w)^2}\Delta w.$$

The relative error (see Art. 33) is given by the equation

$$\frac{\Delta S}{S} = \frac{w}{W-w}\left(\frac{\Delta w}{w} - \frac{\Delta W}{W}\right).$$

EXERCISES.

1. If $u = x^2y$, find approximately the change in u when x changes from 10 to 10.02 and y from 4 to 4.01. Calculate also the change in u when $\Delta x = .002$ and $\Delta y = .001$. In each case compare these approximate changes in u with the actual changes as derived from the original equation. Interpret the results in connection with Fig. 82.

2. Let $z = \sin x \cos y$. If $x = 22°$, $y = 37°$, find the change of z due to the changes $\Delta x = 10'$, $\Delta y = 15'$. Calculate the approximate change and compare the result with the actual change.

3. Let b and h be respectively the breadth and height of a rectangle and A the area; then $A = bh$. Interpret geometrically the approximate equation

$$\Delta A = h\,\Delta b + b\,\Delta h,$$

and show wherein the approximation lies.

4. Find the relative error in the computed area of an ellipse due to errors in the measurement of its semi-axes a and b.

5. Given a triangle having sides a, b, c, and opposite angles A, B, and C. If a side c is determined by measuring two sides a and b and the included angle C, show that the approximate error Δc is given by the equation

$$\Delta c = \Delta a \cos B + \Delta b \cos A + a\,\Delta C \sin B.$$

6. Find the approximate error ΔA when angle A is found by measurement of the three sides.

7. If the area S of the triangle is determined from the three sides, show that

$$\Delta S = \frac{a}{2}\Delta a \cot A + \frac{b}{2}\Delta b \cot B + \frac{c}{2}\Delta c \cot C.$$

8. Derive an expression for ΔS when the area is found from two sides and the included angle.

110. Exact and inexact differentials. In dealing with functions of two or more variables, expressions of the form

$$M\,dx + N\,dy \tag{1}$$

are frequently met, where M and N are functions of x and y. These expressions fall into two classes between which there is an important distinction. It may be possible to find some function $u = f(x, y)$ which upon differentiation gives the given expression. For example,

$$\frac{y}{x}\,dx + \log x\,dy$$

is the result of differentiating $y \log x$. In this case, if we put $u = y \log x$, we have

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y} dy = M\, dx + N\, dy, \tag{2}$$

whence
$$M = \frac{\partial u}{\partial x},\ N = \frac{\partial u}{\partial y}; \tag{3}$$

that is, M and N are the partial derivatives of the given function with respect to the two variables.

Often no function can be found such that the coefficients M and N are its partial derivatives; hence in such cases there is no function whose differentiation gives an expression like (1). When (3) holds, that is, when (1) may be obtained by differentiating some function $u = f(x, y)$, then $M\, dx + N\, dy$ is said to be an *exact differential;* otherwise it is said to be an *inexact differential.*

Whether a given differential is exact or inexact can be readily determined when $u = f(x, y)$ has continuous first and second partial derivatives. If (1) is exact, (3) must be satisfied. From (3) we get by differentiation,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\, \partial y}. \tag{4}$$

It has been shown that when the partial derivatives of u are continuous in both variables, we have

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y};$$

hence the necessary condition that $M\, dx + N\, dy$ be exact, is given by the relation
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{5}$$

It may also be shown that this condition is also sufficient; that is to say, when (5) holds, then some function $u = f(x, y)$ can be found such that (1) is its differential.*

Ex. 1. Given the function

$$e^x y^3\, dx + 3\, e^x y^2\, dy.$$

We have
$$M = e^x y^3, \quad N = 3\, e^x y^2,$$

and
$$\frac{\partial M}{\partial y} = 3\, e^x y^2 = \frac{\partial N}{\partial x}.$$

Hence the given function is an exact differential of $u = e^x y^3$.

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* See Chapter XVIII, Art. 160.

The essential difference between exact and inexact differentials appears when integration is attempted. Suppose first that $M\,dx + N\,dy$ is an exact differential. We may then find some function $u = f(x, y)$, whose differential is the given expression; that is, we may write

$$M\,dx + N\,dy = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \tag{6}$$

$$= \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}\right)dx \tag{7}$$

$$= \frac{du}{dx}\,dx \qquad \text{[from (}b\text{), Art. 106.]} \tag{8}$$

$$= du.$$

Upon integration between any fixed limits, say (x_1, y_1) and (x_2, y_2), we have then

$$\int_{x_1, y_1}^{x_2, y_2} du = u_2 - u_1 = f(x_2, y_2) - f(x_1, y_1). \tag{9}$$

In this case, it will be noticed that the value of the resulting integral depends upon the values of x, y at the initial and final points only. It is not dependent in any way upon the path by which this end point is reached, that is, the manner of variation of x and y between the initial and final points. An example will illustrate this characteristic of exact differentials.

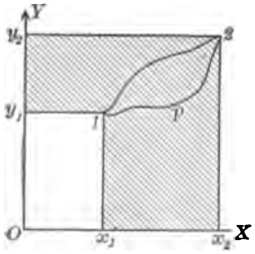

Fig. 83.

Ex. 2. Let $M\,dx + N\,dy = y\,dx + x\,dy.$

Here $du = d(xy)$

and $\int_{x_1, y_1}^{x_2, y_2} du = u_2 - u_1 = x_2y_2 - x_1y_1.$

The integral, which is represented by the shaded area, Fig. 83, is independent of the path p between the points 1 and 2.

Let us suppose now that the expression (1) is not exact. We cannot now replace M and N by the partial derivatives of a function $u = f(x, y)$ with respect to x and y respectively. In order to integrate (1), we must assume some relation between x and y, say $y = F(x)$. We may then write

$$M\,dx + N\,dy = \left(M + N\frac{dy}{dx}\right)dx,$$

which, because of the assumed relation between x and y, is a function of x alone and may be integrated as in functions of a single variable. But the integral

$$\int_{x_1}^{x_2}\left(M + N\frac{dy}{dx}\right)dx$$

depends not only upon the value of x at the initial and final points, but also upon $\frac{dy}{dx}$, that is, upon the path connecting these points.

Ex. 3. Given $M\,dx + N\,dy = y\,dx - x\,dy$. Investigate the integral between the points (0, 0) and (1, 1).

This differential is not exact because

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = -1.$$

If we let $y = x^2$, we have

$$\int_0^1\left(M + N\frac{dy}{dx}\right)dx = -\int_0^1 x^2\,dx = -\tfrac{1}{3}.$$

If, however, we put $y = x^3$, we obtain

$$\int_0^1\left(M + N\frac{dy}{dx}\right)dx = -\int_0^1 2\,x^3\,dx = -\tfrac{1}{2}.$$

Hence in this case, as with all inexact differentials, the integral is dependent not only upon the limits of integration but also upon the variation of x between these limits.

A function which depends upon the coördinates only is often called a *point function.* To distinguish from this a function which depends also upon path of variation, the latter is sometimes called an *imperfect function.* A definite integral of an exact differential is therefore a point function, while a definite integral of an inexact differential is an imperfect function.

It is customary to write

$$du = M\,dx + N\,dy$$

whether $M\,dx + N\,dy$ is exact or inexact. In case $M\,dx + N\,dy$ is exact, u is a point function, and, as in Ex. 2, we may write at once,

$$\int_{x_1,\,y_1}^{x_2,\,y_2} du = u_2 - u_1.$$

If, however, $M\,dx + N\,dy$ is inexact, u is an imperfect function of x and y, and we must write

$$\int_{x_1, y_1}^{x_2, y_2} du = u_{12},$$

where u_{12} is understood as denoting the change of u from the initial to the final condition, whatever that change may be.

The distinction between exact and inexact differentials is further illustrated by the following examples from mechanics and thermodynamics.

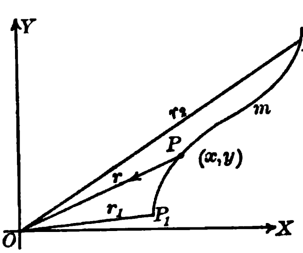

Fig. 84.

Ex. 4. A particle moves in a plane from an initial position P_1, Fig. 84, and traverses some path m. Let F denote the force acting on the particle when in the position P (F is assumed to be in the plane of motion), and let F_x and F_y denote the components of F parallel to the X- and Y-axes, respectively. Then the work of the force F as the particle moves from P_1 to a final position P_2 is

$$W = \int_{P_1}^{P_2} (F_x\,dx + F_y\,dy).^*$$

If the force F at the position P is a function of the coördinates of P, that is, if $F = f(x, y)$, then F_x and F_y are also functions of x and y, and W will or will not depend upon the path m according as $F_x\,dx + F_y\,dy$ is inexact or exact.

Let us suppose that F is directed towards a fixed point, which may be taken as the origin O, and is a function of the distance $OP = r$, that is, $F = \phi(r)$. We have then

$$F_x = F \cos\theta = \phi(r)\frac{x}{r},\; F_y = F \sin\theta = \phi(r)\frac{y}{r},$$

whence
$$W = \int_{x_1, y_1}^{x_2, y_2} \frac{\phi(r)}{r}(x\,dx + y\,dy).$$

Since
$$x^2 + y^2 = r^2, \quad x\,dx + y\,dy = r\,dr,$$

we have
$$W = \int_{r_1}^{r_2} \phi(r)\,dr = \psi(r_2) - \psi(r_1),$$

where
$$\phi(r)\,dr = d\,\psi(r).$$

In this case therefore the differential $F_x\,dx + F_y\,dy$ is exact since it reduces to the differential of a single variable. It follows that the work W depends only upon the initial and final values of r, and not at all upon the path between the points P_1 and P_2.

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* See Ziwet's *Mechanics*, Part III, Art. 71.

In the case of a body moving on a rough surface, the force of friction must be included among the forces acting. The work of this force is given by an expression of the form $\int_{s_1}^{s} \mu Q\,ds$, in which μ denotes the coefficient of friction and Q the pressure between the body and surface. Evidently the work of friction depends upon the path; therefore the total work also depends upon the path, and W cannot be a point function.

Ex. 5. In the expansion of a gas by heat, the heat Q supplied depends upon the changes in the temperature T and volume v of the gas. The equation expressing this dependence is

$$dQ = c\,dT + p\,dv,$$

in which c denotes the specific heat of the gas at constant volume. If we apply the criterion (5), we obtain

$$\frac{\partial c}{\partial v} = 0,\ \frac{\partial p}{\partial T} = \frac{R}{v}\ (\text{since } pv = RT).$$

Hence (5) does not hold, and $c\,dT + p\,dv$ is inexact. This means that the heat Q entering the gas depends upon the way in which T and v vary, that is, upon the path from the initial state (v_1, T_1) to the final state (v_2, T_2). Upon integration, we obtain

$$Q_{12} = c\int_{T_1}^{T_2} dT + \int_{v_1}^{v_2} p\,dv = c(T_2 - T_1) + \int_{v_1}^{v_2} p\,dv.$$

If the relation between p and v is given, $\int_{v_1}^{v_2} p\,dv$ can be evaluated and the heat Q_{12} can be found.

EXERCISES.

1. Determine which of the following differentials are exact.

(a) $e^x \sin y\,dx + e^x \cos y\,dy$. (d) $(y^2 - 2xy - y)\,dx - (x^2 - 2xy + x)\,dy$.

(b) $x^2y^5\,dx + 5x^3y\,dy$. (e) $y \sin 2x\,dx + \sin^2 x\,dy$.

(c) $v^n\,dp + n\,pv^{n-1}\,dv$. (f) $(y^2 - 1)\,dx + x\,dy$.

2. By means of Fig. 83, show the geometrical interpretation of the differential $y\,dx - x\,dy$. Prove from geometrical considerations that this differential cannot be exact.

3. The area under a curve is given by $A = \int_1^2 y\,dx$. Show that A cannot be a point function of the coördinates.

4. By means of the gas equation $pv = RT$ show that while $dQ\ (= c\,dT + p\,dv)$ is inexact, $\frac{dQ}{T}$ is exact.

5. Referring to Fig. 84, find the work of moving the particle from P_1 to P_2 (a) when $F = \frac{k}{r^3}$; (b) when $F = -kr$.

MISCELLANEOUS EXERCISES.

1. Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ for each of the following:

(a) $u = x^2y^3z^5$.   (b) $u = (x^3 + y^3 + z^3)^{\frac{1}{2}}$.

(c) $u = x^3e^{2y}\cos z$.   (d) $u = \arc\tan\frac{x+y}{z}$.

2. Find D_xy for the following functions:

(a) $x^3y^2 - 4\,xy^4 + 3\,y^5 = 0$.

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$.

(c) $ax^2 + by^2 + 2\,hxy + 2\,ex + 2\,fy + g = 0$.

(d) $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

3. Find $\frac{du}{dx}$ in the following cases:

(a) $u = \sqrt{x^2 + y^2}$, $y = \arc\tan x$.

(b) $u = x^3e^x \arc\sin x$

(c) $u = \arc\tan\frac{y}{z}$, $y = e^{-x}$, $z = \cos x$.

(d) $u = e^{ax}(y - z)$, $y = a\sin x$, $z = \cos x$.

4. Form $\frac{\partial^3 u}{\partial x\,\partial y^2}$, $\frac{\partial^3 u}{\partial x^2\partial y}$, and $\frac{\partial^3 u}{\partial y^3}$ for each of the following:

(a) $u = \sin e^{xy}$.   (d) $u = e^x\cos y$.

(b) $u = x^3 - 5\,x^2y + 3\,y^4$.   (e) $u = \sin x^2 + \cos xy$.

(c) $u = \cos(ax^m + by^n)$.   (f) $u = \frac{x}{a + bx^2 + cy^2}$.

5. Show that the relation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ holds for the following functions:

$$V = \log(x^2 + y^2), \qquad V = \arc\tan\frac{y}{x}.$$

6. If $V = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, show that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$.

7. The potential V at a point P, in a straight line due to the attraction of a mass at an external point C, is

$$V = k\log\frac{y + \sqrt{x^2 + y^2}}{x}.$$

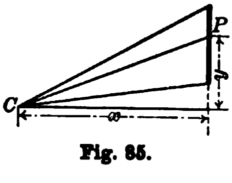

Fig. 85.

Find $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$.

8. Find from van der Waals' equation

$$\left(p + \frac{a}{v^2}\right)(v - b) = BT$$

the partial derivatives $\frac{\partial p}{\partial T}$ and $\frac{\partial p}{\partial v}$. Give physical interpretations to these derivatives.

9. From the perfect gas equation $pv = RT$, find the derivatives $\frac{\partial p}{\partial v}$, $\frac{\partial p}{\partial T}$, $\frac{\partial v}{\partial T}$, and give physical interpretations.

10. At the point $x = 2$, $y = -3$, on the ellipsoid $\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{16} = 1$, find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Draw a figure and interpret the results geometrically.

11. If z is a homogeneous function of x and y of degree n, prove *Euler's theorem*, viz.:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

12. Verify Euler's theorem for the following functions:

(a) $z = (x^2 + y^2)\sqrt{xy}$. (c) $z = \text{arc} \tan xy$.

(b) $z = x^3 - 3xy^2 + 2y^3$. (d) $z = \text{arc} \cos \frac{x}{y} + \text{arc} \tan \frac{y}{x}$.

13. Differentiate both members of (*a*), Art. 106, with respect to t, and derive the following formula for the second total derivative:

$$\frac{d^2z}{dt^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}.$$

14. By taking x itself as the auxiliary variable t, derive from the formula of Ex. 13 a formula for the second derivative $\frac{d^2z}{dx^2}$. Use this formula to find $\frac{d^2z}{dx^2}$ from the equation

$$z = x^2 - 2xy.$$

15. If $z = f(y + ax)$, show that

$$\text{(a)}\ \frac{\partial z}{\partial x} = a\frac{\partial z}{\partial y};\qquad \text{(b)}\ \frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}.$$

16. From the transformation equations $x = \rho \cos \theta$, $y = \rho \sin \theta$, derive the relation

$$x\,dy - y\,dx = \rho^2\,d\theta.$$

17. A point moves on the surface $z = x^2 + 2y^2$ in a vertical plane which includes the Z-axis and makes an angle of 30° with the XZ-plane. If the X-component of the point's velocity is 3 units when $x = 2$, find the Y- and Z-components of the velocity.

18. Derive an approximate value for the error in computing the volume V of a cylinder from the measured height h and base radius r. Find also an expression for the relative error.

19. The specific heat of a substance by the method of mixtures is given by the formula

$$C = \frac{m_1 h_1}{m_2 h_2},$$

in which m_1 and m_2 are two masses, and h_1 and h_2 two temperature differences. All four quantities are determined by measurement. Derive an expression for the error of C due to errors of measurement. Also derive an expression for the relative error.

20. In Ex. 19, with the following data calculate ΔC and $\frac{\Delta C}{C}$

$$m_1 = 0.744 \text{ lb.} \qquad h_1 = 7.767°.$$
$$m_2 = 22.337 \text{ lb.} \qquad h_2 = 8.244°.$$
$$\Delta m_1 = \Delta m_2 = .01 \text{ lb.} \qquad \Delta h_1 = \Delta h_2 = .02°.$$

21. The expression

$$\frac{hx + c(1 - x)}{T}\, dT + \frac{r}{T}\, dx,$$

which applies to a mixture of liquid and vapor, is known to be an exact differential; h and c are specific heats, functions of the temperature T only; r is the latent heat of vaporization, also a function of T; and x is the percentage of vapor in the mixture. By the use of the criterion for an exact differential derive the relation

$$h - c = \frac{dr}{dT} - \frac{r}{T}.$$

22. Find a function u of x and y that satisfies the relation

$$\text{(a)}\ \frac{\partial^2 u}{\partial x\, \partial y} = x^2 + y^2; \qquad \text{(b)}\ \frac{\partial^2 u}{\partial x\, \partial y} = x^3 \cos y.$$

CHAPTER XIII

MULTIPLE INTEGRALS. GEOMETRICAL APPLICATIONS

111. Plane areas by double integration, rectangular coördinates. In Art. 80 was shown the connection between a definite integral and the sum of an infinitely large number of indefinitely small terms, and in Art. 105, the elementary properties of multiple integrals were defined. In the present article, we shall apply the results of the previous discussion to determine plane areas. In doing this, we shall make free use of the results already obtained.

Let a given plane area be bounded by the two curves $y = f(x)$ and $y = F(x)$ (Fig. 86). Circumscribe the bounding curves by a rectangle whose sides are parallel to the axes of coördinates. Divide AB into n and CD into m equal parts, denoting each of the equal divisions by Δx, Δy, respectively. The given rectangle is thus divided into elementary rectangles of area $\Delta x \, \Delta y$. The area of the strip PQ is given by

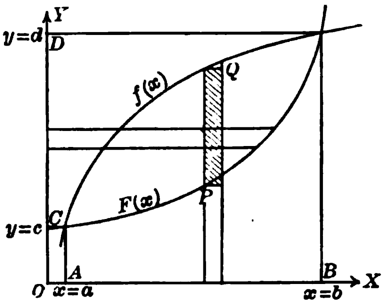

Fig. 86.

$$\Delta x \cdot \underset{\Delta y \doteq 0}{L} \sum_{F(x)}^{f(x)} \Delta y = \Delta x \int_{F(x)}^{f(x)} dy.$$

If now we sum up all of the strips between the limits $x = a$ and $x = b$, we have

$$\sum_{a}^{b} \Delta x \int_{F(x)}^{f(x)} dy.$$

The limit of this sum as $\Delta x \doteq 0$ is, by Art. 82, the required area. However, this limit is simply the definite integral taken from $x = a$ to $x = b$ of the function $\int_{F(x)}^{f(x)} dy$. Hence, denoting the required area by A, we have upon passing to the definite integral,

$$A = \int_a^b \int_{F(x)}^{f(x)} dy\, dx. \tag{1}$$

We might equally well have summed up the elements of area in the reverse order, namely, first with respect to x and then with respect to y. In this case, we should have obtained

$$A = \int_c^d \int_{\phi(y)}^{\psi(y)} dx\, dy, \tag{2}$$

where $\phi(y)$ and $\psi(y)$ are the inverse functions of $F(x)$, $f(x)$, respectively.

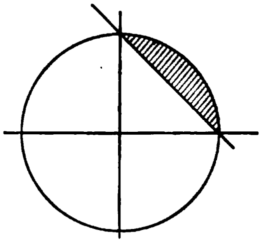

Fig. 87.

Ex. 1. Find the area between the circle $x^2+y^2=a^2$ and the line $x + y = a$. See Fig. 87.

The coördinates of the points of intersection are $(0, a)$ and $(a, 0)$. The lower limit of the y-integral is the value of y found from the equation $x + y = a$, namely, $y = a - x$; and the upper limit is the value of y determined from the equation $x^2 + y^2 = a^2$, namely, $y = \sqrt{a^2 - x^2}$. Hence we have

$$\begin{aligned} A &= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\, dx \\ &= \int_0^a \left[\sqrt{a^2 - x^2} - (a - x)\right] dx \\ &= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \text{arc sin } \frac{x}{a} - ax + \frac{x^2}{2}\right]_0^a \\ &= \frac{\pi - 2}{4} a^2. \end{aligned}$$

In some cases it is necessary to break up the integration into two or more parts, as illustrated by the following example.

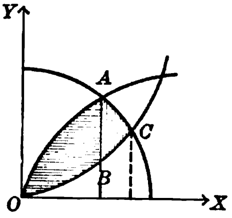

Fig. 88.

Ex. 2. Find the area bounded by the curves $x^2 + y^2 = 25$, $y^2 = \frac{16}{3} x$, and $y = \frac{3}{16} x^2$. See Fig. 88.

The circle and the first parabola intersect at the point A whose coördinates are (3, 4), and the second parabola intersects at the point C whose coördinates are (4, 3). From O to A, the equation $y^2 = \frac{16}{3} x$ gives the upper y-limit, but from A to C, this limit is determined from the equation of the circle, $x^2 + y^2 = 25$. The equation $y = \frac{3}{16} x^2$ of the lower curve OC gives the lower y-limit throughout. Hence we have

$$A = \int_0^3 \int_{\frac{3}{16}x^2}^{4\sqrt{\frac{x}{3}}} dy\, dx + \int_3^4 \int_{\frac{3}{16}x^2}^{\sqrt{25-x^2}} dy\, dx = 7.55.$$

EXERCISES.

1. Find by double integration the area of a parallelogram with one side in the X-axis.

2. Find by double integration the area of a right triangle with a short side in the X-axis.

3. Find by double integration the area between the parabolas $y^2 = 8x$ and $x^2 = 8y$.

4. Find the area between the circle $x^2 + y^2 = a^2$ and the line $y = b$, $b < a$.

5. Find the area between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.

6. Find the area between the curve $y = x^3$ and the Y-axis from the origin to $y = 8$.

7. Find the area bounded by the parabolas $y^2 = \frac{16}{3}x$, $y^2 = \frac{9}{4}x$ and the circle $x^2 + y^2 = 25$. Consider the first quadrant only.

8. Find the area between the circle $x^2 + y^2 = 25$ and the line $x + y = 7$.

9. Find the area bounded by the curves $x^2 + y^2 = 169$, $y^2 = \frac{144}{125}x^3$, and $5x - 12y = 0$, in first quadrant.

10. Find by double integration the area of the circle

$$(x - a)^2 + (y - b)^2 = c^2.$$

11. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

112. Plane areas by double integration, polar coördinates. It is sometimes convenient to choose a polar element of area rather than the rectangular element $\Delta y\,\Delta x$. To derive this polar element, we proceed as follows.

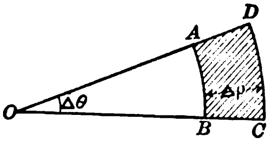

Fig. 89.

Let the given element of area be $ABCD$, Fig. 89, bounded by the two radii OC, OD, and two circular arcs having their common center at O. Let the polar coördinates of B be (ρ, θ). From elementary geometry, we have

$$\text{sector } AOB = \tfrac{1}{2}\rho^2\Delta\theta, \tag{1}$$

$$\text{sector } DOC = \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta. \tag{2}$$

Hence

$$\begin{aligned}\Delta A = ABCD &= \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta - \tfrac{1}{2}\rho^2\Delta\theta \\ &= (\rho + \tfrac{1}{2}\Delta\rho)\,\Delta\theta\,\Delta\rho. \end{aligned}\tag{3}$$

Using this polar element, we may find the area between two curves $\rho = F(\theta)$, $\rho = f(\theta)$, Fig. 90, as follows: First keep $\Delta\theta$ constant and sum the area elements with respect to ρ. The result of this summation is an area of the type $PQSR$, the expression for which is

$$\Delta\theta \cdot \underset{\Delta\rho \doteq 0}{L} \sum_{OQ}^{OP} (\rho + \tfrac{1}{2}\Delta\rho)\Delta\rho$$

$$= \Delta\theta \cdot \underset{\Delta\rho \doteq 0}{L} \sum_{OQ}^{OP} \rho\Delta\rho$$

$$= \Delta\,\theta \int_{OQ}^{OP} \rho\, d\rho.$$

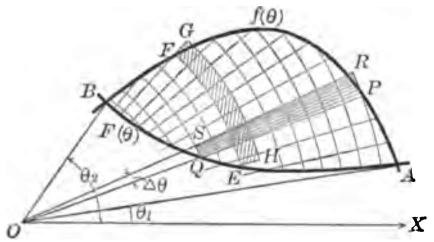

Fig. 90.

If now we sum with respect to θ, we get the sum of the wedge-like slices, and the limit of this sum is the required area. We have therefore

$$A = \underset{\Delta\theta \doteq 0}{L} \sum_{\theta_1}^{\theta_2} \Delta\theta \cdot \int_{OQ}^{OP} \rho\, d\rho = \int_{\theta_1}^{\theta_2}\int_{OQ}^{OP} \rho\, d\rho d\theta.$$

Replacing OQ and OP by $F(\theta)$ and $f(\theta)$ respectively, we obtain the formula

$$A = \int_{\theta_1}^{\theta_2}\int_{F(\theta)}^{f(\theta)} \rho\, d\rho\, d\theta. \tag{4}$$

The area included between the curve $\rho = f(\theta)$ and two radii, as OA and OB, is obtained from (4) by making $F(\theta) = 0$.

The required area may also be obtained as follows: Summing first with respect to θ, keeping $\Delta\rho$ constant, we obtain a segment of a circular ring of the type $EFGH$. A second summation with respect to ρ gives the sum of such ring segments, the limit of which sum is the area A. The resulting formula is

$$A = \int_{\rho_1}^{\rho_2}\int_{\phi(\rho)}^{\psi(\rho)} \rho\, d\theta\, d\rho, \tag{5}$$

where $\phi(\rho)$ and $\psi(\rho)$ are the inverse functions of $F(\theta)$ and $f(\theta)$ respectively.

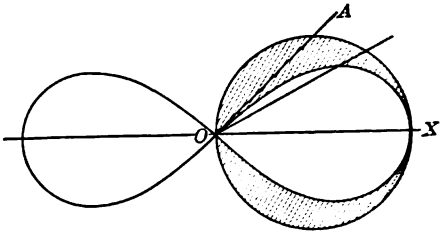

Fig. 91.

Ex. Find the area between the circle $\rho = \cos\theta$ and one loop of the lemniscate $\rho^2 = \cos 2\,\theta$. Fig. 91.

If formula (4) is used, the area must be taken in two parts. The upper limit for both integrations is determined from the equation of the circle $\rho = \cos\theta$. For values of θ between 0 and $\frac{\pi}{4}$, the lower limit of the ρ-integration is obtained from the equation of the lemniscate $\rho^2 = \cos 2\theta$. Since, however, no part of this loop of the lemniscate lies to the left of the line OA, the lower limit for the ρ-integration for values of θ greater than $\frac{\pi}{4}$ is 0. Hence we have

$$A = 2\int_0^{\frac{\pi}{4}}\int_{\sqrt{\cos 2\theta}}^{\cos\theta} \rho\, d\rho\, d\theta + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta} \rho\, d\rho\, d\theta = \frac{\pi-2}{4}.$$

EXERCISES.

1. Find by double integration the area of the circle $\rho = a$.

2. Find the area between the cardioid $\rho = a(1 - \cos\theta)$ and the circle $\rho = -a\cos\theta$.

3. Find the area between two circles tangent internally and having radii r_1 and r_2 respectively.

4. Find the entire area of the cardioid $\rho = a(1 - \cos\theta)$.

5. Find the area of one loop of the lemniscate $\rho^2 = a^2\cos 2\theta$.

6. Find the area between the circle $\rho = \cos\theta$ and one loop of the lemniscate $\rho^2 = \cos 2\theta$ by the use of formula (5).

7. Find the areas between the cardioid $\rho = a(1 - \cos\theta)$ and the circle $\rho = a$. (The shaded areas $OBAD$ and $BDCB$, Fig. 92.)

8. Work Ex. 3 by formula (5).

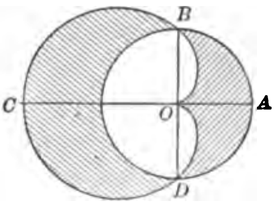

Fig. 92.

113. Volumes by triple integration, rectangular coördinates. Consider the volume bounded by the coördinate planes and any surface whose equation is $z = f(x,y)$, where $f(x, y)$ is a continuous function. Through two neighboring points P and Q (Fig. 93) within the solid let planes be passed parallel to the three coördinate planes. The six planes will inclose a rectangular parallelopiped PQ whose volume is $\Delta z\, \Delta y\, \Delta x$; for if we assume the coördinates of P and Q to be respectively (x, y, z) and $(x + \Delta x, y + \Delta y, z + \Delta z)$, the edges of the parallelopiped are Δx, Δy, and Δz, respectively.

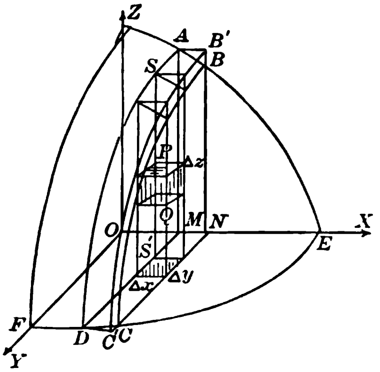

Fig. 93.

We first take the sum of these elementary parallelopipeds with edges lying along the line $S'PS$. With x, y, Δx, and Δy constant, the limit of this sum as Δz is made to approach zero is the volume of the prism whose base is $\Delta x\, \Delta y$ and whose altitude is $S'S$. Now with x and Δx constant, we take the sum of the prisms lying between the planes AMD and BNC. By an extension of the method of Art. 82, it follows that as Δy is made to approach zero, this sum approaches the volume of the cylindrical slice $AB'C'DMN$. Finally, we take the sum of the slices parallel to the YZ-plane. As Δx approaches zero, the volume of the cylindrical slice approaches that of the actual slice $ABCDMN$. Hence the limit of the sum of these slices as Δx approaches zero, gives the volume of the solid. We have therefore

$$\text{Volume of prism} = \Delta y\, \Delta x \underset{\Delta z \doteq 0}{L} \sum_{0}^{SS'} \Delta z = \Delta y\, \Delta x \int_0^{SS'} dz.$$

$$\text{Volume of cylindrical slice} = \Delta x \underset{\Delta y \doteq 0}{L} \sum_{0}^{MD} \Delta y \left(\int_0^{SS'} dz \right)$$

$$= \Delta x \int_0^{MD} \int_0^{SS'} dz\, dy.$$

$$\text{Volume of solid} = \underset{\Delta x \doteq 0}{L} \sum_{0}^{OE} \Delta x \left(\int_0^{MD} \int_0^{SS'} dy\, dz \right)$$

$$= \int_0^{OE} \int_0^{MD} \int_0^{SS'} dz\, dy\, dx. \tag{1}$$

Hence the volume is obtained by a triple integration, provided the limits of integration are properly chosen.

The first summation, namely, that with respect to z, extends from zero to SS', which is the value of the ordinate of a point on the surface. The upper limit is, therefore, a variable which is given by $z = f(x, y)$, where x, y are the coördinates of the point S'. The second summation is taken from zero to MD, and hence the limits are zero, and the value of y on the curve FDE, that is, the value of y determined from the equation $f(x, y) = 0$. Let this value be $y = \phi(x)$. Finally, the integration which gives the sum of all the slices between O and E has for its limits zero and the constant OE ($= a$, say). We may therefore write equation (1) in the form

$$V = \int_0^{a} \int_0^{y=\phi(x)} \int_0^{z=f(x,\, y)} dz\, dy\, dx. \tag{2}$$

It will be observed that the limits for the second and third integration are the same as those that would be used in finding the area *OEF*, i.e., the projection of the given solid on the *XY*-plane, by means of the double integral $\int\int dy\,dx$.

If the given solid is not bounded by the coördinate planes, the lower limits will not be zero, as in (1) and (2); they can be readily determined, however, in the same manner as the upper limits.

Ex. Find the volume inclosed by the surface $\frac{x^2}{a} + \frac{y^2}{b} = 2z$ and the plane $z = c$.

Since the solid is symmetrical with respect to the *XZ*- and *YZ*-planes, we may find the volume of the part between these planes and multiply by 4. A line through $P(x, y, z)$ parallel to OZ cuts the bounding plane in a point for which $z = c$, and the bounding surface in a point for which $z = \frac{1}{2}\left(\frac{x^2}{a} + \frac{y^2}{b}\right)$. These values of z are therefore the upper and lower limits for the first integration. A plane section through P parallel to the plane YZ has for its perimeter the parabola $y^2 = 2bz - \frac{b}{a}x^2$. The extreme value of y, that for which $z = c$, is

$$y = \sqrt{2bc - \frac{b}{a}x^2},$$

and this is therefore the upper limit for the second integration. Finally, the projection of the solid on the plane XY is bounded by the ellipse

$$\frac{x^2}{a} + \frac{y^2}{b} = 2c,$$

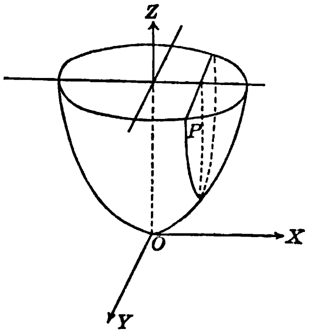

Fig. 94.

and the extreme value of x, corresponding to $y = 0$, is $\sqrt{2ac}$. Hence we have

$$V = 4\int_0^{\sqrt{2ac}}\int_0^{\sqrt{2bc - \frac{b}{a}x^2}}\int_{\frac{1}{2}\left(\frac{x^2}{a} + \frac{y^2}{b}\right)}^{c} dz\,dy\,dx = c^2\sqrt{ab}.$$

EXERCISES.

1. Determine the limits of integration for the triple integral $\int\int\int dz\,dy\,dx$ required in finding the volume of the pyramid bounded by the coördinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Draw the figure.

2. The cone whose equation is $y^2 + z^2 = c^2x^2$ has the X-axis as its axis. Determine the limits of integration when the volume of the cone is found from the triple integral $\int\int\int dz\, dy\, dx$. Also when the volume is found from the triple integral $\int\int\int dx\, dy\, dz$, the x-integration being performed first. Let h denote the altitude.

3. Find the volumes in Exs. 1 and 2.

4. Find by triple integration the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

5. Find the volume inclosed by the surface

$$x^2y^2 + c^2z^2 = a^2y^2,$$

and the planes $y = 0$ and $y = c$.

6. Show that the volume generated by revolving a plane about the X-axis may be found from the formula

$$V = 2\pi \int\int y\, dy\, dx.$$

7. Find the entire volume bounded by the surface whose equation is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

8. Find the volume of the wedge cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$, and $z = x \tan\beta$.

114. Volumes by triple integration, polar coördinates. Let ρ, θ, ϕ, denote the coördinates of any point P within a solid, where, as usual, ρ is the distance OP (see Fig. 95), θ is the angle ZOP which OP makes with the Z-axis, and ϕ is the angle XOP' which the projection of OP on the XY-plane makes with the X-axis. Through P suppose three surfaces passed: (1) the surface of a sphere with radius OP and O as a center; (2) a conical surface produced by revolving OP about OZ as an axis; (3) a plane surface passed through OP and OZ. Now let a second spherical surface be passed through P_1, a second conical surface through S by the rotation of OSS_1 about OZ, and a second plane surface through OZ and OQ. The six surfaces inclose a solid element $PQRSS_1R_1Q_1P_1$ having the two spherical surfaces $PQRS$ and $P_1Q_1R_1S_1$, the two conical surfaces

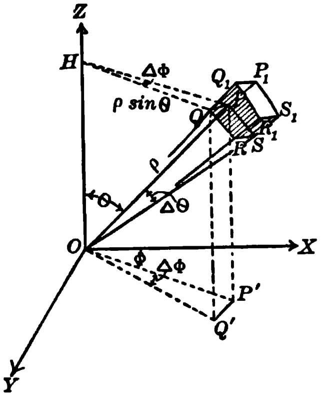

Fig. 95.

PP_1Q_1Q and SS_1R_1R, and the plane surfaces PP_1S_1S and QQ_1R_1R. By means of such sets of surfaces the entire solid may be divided into elements of this type.

Let $PP_1 = \Delta\rho$, angle $POS = \Delta\theta$, and angle $P'OQ'$ = angle $PHQ = \Delta\phi$. Then the sides of the given element of volume are $PP' = \Delta\rho$, $PQ = PH\ \Delta\phi = \rho \sin\theta\ \Delta\phi$, and $PS = \rho\ \Delta\theta$. Consequently, the volume of this element is $\rho^2 \sin\theta\ \Delta\theta\ \Delta\phi\ \Delta\rho$, plus other terms which vanish when we pass to the limit.* Therefore, denoting the required volume by V, we may write,

$$V = \underset{\Delta\theta \doteq 0}{L}\ \underset{\Delta\phi \doteq 0}{L}\ \underset{\Delta\rho \doteq 0}{L} \sum\sum\sum \rho^2 \sin\theta\ \Delta\rho\Delta\phi\Delta\theta,$$

or

$$V = \int\int\int \rho^2 \sin\theta\ d\rho\ d\phi\ d\theta, \tag{1}$$

each integral being taken between the limits required by the conditions of the problem under consideration. This summation of the elements of volume can be effected in any order so long as the given volume is continuous. The method of determining the limits of integration is illustrated by the following example:

Ex. 1. Find the volume included by the surface $\rho = a \sin^2\theta \cos\phi$.

This volume lies entirely to the right of a tangent plane through the origin perpendicular to the initial line. The limits for ϕ are, therefore, $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and the entire volume will be included if θ is given the limits 0 and π (see Fig. 95). The limits of ρ are, of course, 0 and $a \sin^2\theta \cos\phi$. Hence we have

$$V = \int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a \sin^2\theta \cos\phi} \rho^2 \sin\theta\ d\rho\ d\phi\ d\theta$$

$$= \frac{a^3}{3} \int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7\theta \cos^3\phi\ d\phi\ d\theta = \frac{128}{315} a^3.$$

For a solid of revolution formula (1) may be simplified as follows: Taking the Z-axis as the axis of revolution, the limits of the

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* The exact expression for the volume element is

$$\rho^2 \sin\theta\ \Delta\rho\ \Delta\theta\ \Delta\phi \left[\frac{\sin\left(\theta + \frac{\Delta\theta}{2}\right)}{\sin\theta}\right]\left(\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right)\left(1 + \frac{\Delta\rho}{\rho} + \frac{\Delta\rho^2}{3\rho^2}\right).$$

It will be seen that as $\Delta\theta \doteq 0$ the factors in the first two parentheses approach 1, and as $\Delta\rho \doteq 0$, the last factor approaches 1.

integration with respect to ϕ are clearly 0 and 2π; hence (1) becomes

$$V = 2\pi \int\int \rho^2 \sin\theta \, d\rho \, d\theta. \tag{2}$$

The limits of integration for ρ and θ are precisely those that would be employed in finding the area of the plane figure revolved.

Ex. 2. Find the volume generated by revolving the cardioid $\rho = a(1 - \cos\theta)$ about the initial line.

We have in this case

$$V = 2\pi \int_0^\pi \int_0^{a(1-\cos\theta)} \rho^2 \sin\theta \, d\rho \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^\pi (1 - \cos\theta)^3 \sin\theta \, d\theta = \frac{8}{3}\pi a^3.$$

While in most cases the rectangular and polar volume elements previously deduced are the most convenient to use, it is sometimes possible to choose other volume elements that lead to simpler integrations. See Ex. 12 at the end of this chapter.

EXERCISES.

1. Find by polar coördinates the volume of a sphere (a) when the origin is taken at the center; (b) when the origin is taken at the end of a diameter.

2. By passing to polar coördinates find the entire volume of the solid included by the surface

$$(x^2 + y^2 + z^2)^2 = axyz.$$

3. In the same manner find the volume included by the surface

$$(x^2 + y^2 + z^2)^3 = 27\, a^3xyz.$$

4. Find the volume obtained by revolving the lemniscate $\rho^2 = a^2 \cos 2\theta$ about the initial line.

5. Show the different types of solid elements produced when the summation is effected in the following orders:

(a) with respect to ρ, then θ, then ϕ;
(b) with respect to ρ, then ϕ, then θ;
(c) with respect to θ, then ϕ, then ρ.

115. Equation of tangent plane. Let the equation of a surface be given in the explicit form

$$z = \phi(x, y), \tag{1}$$

and let (x', y', z') be the coördinates of some point P on the surface, Fig. 96. The plane $x = x'$ cuts the surface in a curve whose tangent t at the given point P has the slope

$$\phi_y'(x', y') = \frac{\partial y}{\partial z}\Big]_{y=y'}.$$

For the sake of brevity, we write $\frac{\partial z}{\partial y'}$ for $\phi_y'(x', y')$. Also the plane $y = y'$ cuts the surface in a curve whose tangent at P has the slope $\frac{\partial z}{\partial x'}$. Hence, we may write the equations of the two tangents at P in the form

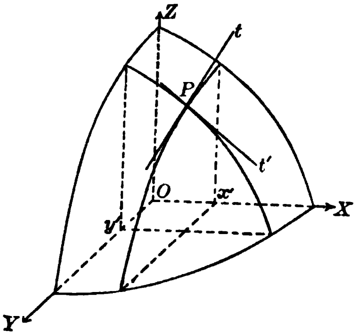

Fig. 96.

$$z - z' = \frac{\partial z}{\partial y'}(y - y'), \tag{2}$$

$$z - z' = \frac{\partial z}{\partial x'}(x - x'). \tag{3}$$

Let the equation of the tangent plane containing the tangents t and t' at the point P be

$$Ax + By + Cz + D = 0; \tag{4}$$

then the following conditions must be fulfilled:

(a) The plane must pass through P, that is,

$$Ax' + By' + Cz' + D = 0. \tag{5}$$

(b) When $x = x'$, (2) must be satisfied.

(c) When $y = y'$, (3) must be satisfied.

Combining (4) and (5), we get

$$\frac{A}{C}(x - x') + \frac{B}{C}(y - y') + (z - z') = 0. \tag{6}$$

When $x = x'$, we have from (6),

$$z - z' = -\frac{B}{C}(y - y'),$$

and from (2), $$z - z' = \frac{\partial z}{\partial y'}(y - y').$$

Hence
$$-\frac{B}{C}=\frac{\partial z}{\partial y'};$$
and similarly,
$$-\frac{A}{C}=\frac{\partial z}{\partial x'}.$$

Substituting these values in (6), we get finally,

$$z-z'=\frac{\partial z}{\partial y'}(y-y')+\frac{\partial z}{\partial x'}(x-x'), \tag{7}$$

which is the equation of the tangent plane at the point $P\,(x', y', z')$.

If the equation of the surface is given in the implicit form

$$f(x, y, z) = 0,$$

the equation of the tangent plane takes the form

$$(x-x')\frac{\partial f}{\partial x'}+(y-y')\frac{\partial f}{\partial y'}+(z-z')\frac{\partial f}{\partial z'}=0. \tag{8}$$

This form may be derived from that given in (7) by the means of the relations

$$\frac{\partial z}{\partial y'}=-\frac{\frac{\partial f}{\partial y'}}{\frac{\partial f}{\partial z'}};\quad \frac{\partial z}{\partial x'}=-\frac{\frac{\partial f}{\partial x'}}{\frac{\partial f}{\partial z'}}.$$

116. Equations of a normal. The normal to the surface at the point $P\,(x', y', z')$ must be perpendicular to the tangents t and t' through P. Hence from (2) and (3) of the preceding article, the normal must have the equations

$$z-z'=-\frac{y-y'}{\frac{\partial z}{\partial y'}}=-\frac{x-x'}{\frac{\partial z}{\partial x'}}. \tag{1}$$

If the equation of the surface is in the implicit form, the equations of the normal may be written

$$\frac{x-x'}{\frac{\partial f}{\partial x'}}=\frac{y-y'}{\frac{\partial f}{\partial y'}}=\frac{z-z'}{\frac{\partial f}{\partial z'}}. \tag{2}$$

In equations (2) the denominators are proportional to the direction cosines of the normal.

The angle which the tangent plane at P makes with one of the coördinate planes, say the XY-plane, is evidently the same as the angle between two intersecting lines perpendicular to these planes. Any perpendicular to the tangent plane has direction cosines proportional to

$$\frac{\partial f}{\partial x'},\quad \frac{\partial f}{\partial y'},\quad \frac{\partial f}{\partial z'},$$

while the Z-axis, perpendicular to the XY-plane, has the direction cosines

$$0,\ 0,\ 1.$$

Let γ denote the angle between these perpendiculars; then we have from solid analytic geometry,*

$$\cos\gamma = \frac{0\cdot\frac{\partial f}{\partial x'} + 0\cdot\frac{\partial f}{\partial y'} + 1\cdot\frac{\partial f}{\partial z'}}{\sqrt{\left(\frac{\partial f}{\partial x'}\right)^2+\left(\frac{\partial f}{\partial y'}\right)^2+\left(\frac{\partial f}{\partial z'}\right)^2}} = \frac{\frac{\partial f}{\partial z'}}{\sqrt{\left(\frac{\partial f}{\partial x'}\right)^2+\left(\frac{\partial f}{\partial y'}\right)^2+\left(\frac{\partial f}{\partial z'}\right)^2}}. \tag{3}$$

The angle between the tangent plane and the XZ- or YZ- plane is found in a similar manner.

Ex. Given the surface whose equation is $xy - c^2z^2 = 0$, to find the tangent plane at the point $\left(1,\ 1,\ \frac{1}{c}\right)$.

We have

$$\frac{\partial f}{\partial x} = y,\quad \frac{\partial f}{\partial y} = x,\quad \frac{\partial f}{\partial z} = -2c^2z,$$

whence

$$\frac{\partial f}{\partial x'} = 1,\quad \frac{\partial f}{\partial y'} = 1,\quad \frac{\partial f}{\partial z'} = -2c.$$

The equation of the tangent plane is therefore

$$(x-1) + (y-1) - 2c\left(z - \frac{1}{c}\right) = 0,$$

or

$$x + y - 2cz = 0.$$

The equations of the normal at the given point are

$$x - 1 = y - 1 = -\frac{z - \frac{1}{c}}{2c}.$$

The angle which the tangent plane makes with the XY-plane is given by the equation

$$\cos\gamma = -\frac{2c}{\sqrt{2+4c^2}}.$$

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* See Chas. Smith's *Solid Geometry*, Art. 24.

EXERCISES.

1. Find the equations of the plane tangent to the surface

$$ax^2 + by^2 + cz^2 + d = 0,$$

at the point (x_0, y_0, z_0).

Find the equations of the normal at the same point.

2. Find the equation of the tangent plane and the equations of the normal to the ellipsoid

$$\frac{x^2}{36} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

at the point $(5, -2, \frac{1}{2}\sqrt{2})$.

3. In Ex. 2, find the angle which the tangent plane makes with each of the coördinate planes.

4. Find the equation of the tangent plane to the surface $xy = 4z$ at the point $(-2, 6, -3)$. Find the angle which this plane makes with the XY-plane.

117. Area of any surface. By double integration we may, in general, find the area of any surface whose equation is known. Let the equation be

$$f(x, y, z) = 0,$$

and let (x_1, y_1, z_1) be the coördinates of some point S, Fig. 97, on the surface. The coördinates of the point U may then be taken as $x + \Delta x$, $y + \Delta y$, $z + \Delta z$. Suppose planes passed through S and U parallel to the planes XZ and YZ. These planes will cut from the surface the curved quadrilateral $STUV$ and from the plane XY the rectangle $S'T'U'V'$, which is the projection of $STUV$. Conceive a tangent plane passed through the point S. If the ordinates TT', UU', and VV' are prolonged to intersect this plane, a parallelogram $ST''U''V''$ (see Fig. 98) will be formed on this plane. The area of the projection $S'T'U'V'$ is $\Delta y \, \Delta x$, and if γ denotes the angle which the tangent plane at S makes with the plane XY, the area of the parallelogram $ST''U''V''$ on the tangent plane is the product $\sec \gamma \, \Delta y \, \Delta x$.

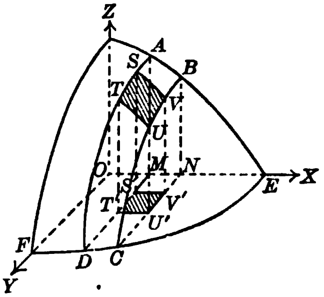

Fig. 97.

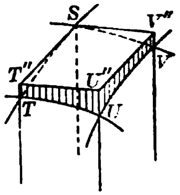

Fig. 98.

With x and Δx remaining constant, suppose Δy to approach zero.

Evidently the point T'' on the tangent plane will approach the point T on the surface, and the limit

$$\underset{\Delta y \doteq 0}{L} \sum \sec \gamma \, \Delta y \, \Delta x = \Delta x \underset{\Delta y \doteq 0}{L} \sum \sec \gamma \, \Delta y \tag{1}$$

of the sum of the parallelograms is the area of a surface included between the planes AMD and BNC. The edge of this surface coincides with the line ASD; the other edge, however, does not, in general, coincide with line BVC, but is the locus of the point V'' on the tangent plane. Now as Δx is made to approach zero, the points V'' approach the points V on the surface, and the area given by (1) approaches the true area $ABCD$ intercepted between the two planes. Hence the limit of the sum of terms of the type shown by (1), as $\Delta x \doteq 0$, is the area of the surface in question; that is,

$$\begin{aligned} A &= \underset{\Delta x \doteq 0}{L} \sum \Delta x \Big(\underset{\Delta y \doteq 0}{L} \sum \sec \gamma \, \Delta y \Big) \\ &= \int_0^{OE} \Big(\int_0^{MD} \sec \gamma \, dy \Big) \, dx \\ &= \int_0^{OE} \int_0^{MD} \sec \gamma \, dy \, dx. \end{aligned} \tag{2}$$

The limits of integration must be such as to include the entire surface. Referring to the figure, it is evident that the y-limits for the curved area $ABCD$ are precisely the same as those for the plane area $MNCD$, and that the x-limits are those that would be used in summing the slices $MNCD$ to obtain the area OEF, that is, the projection of the given surface on the plane XY. Hence in finding the area of the surface

$$f(x, y, z) = 0 \tag{3}$$

by the double integral $\int\int \sec \gamma \, dy \, dx$, we use the same limits as in finding the area of the projection of the surface on the XY-plane by means of the double integral $\int\int dy \, dx$. As shown in the figure, the surface is limited to the positive octant. When so limited, each of the lower limits of integration is zero, while the upper limit when integrating with respect to y is a function of x, say $\phi(x)$, and is obtained by putting $z = 0$ in (3) and solving the result for y. In integrating with respect to x, the upper limit is a constant OE $(= b$, say). The surface may not, however, be bounded by the coördinate planes; hence the lower limits of integration will, in general, be different from zero, say $y = \psi(x)$ in the one case and $x = a$ in the other.

Before the integration can be effected, the factor $\sec\gamma$ must be expressed in terms of the variables x and y. Let α, β, γ denote the angles which the tangent plane at S (x, y, z) makes with the planes YZ, ZX, and XY, respectively; then from Art. 116, we have

$$\cos\alpha = \frac{\frac{\partial f}{\partial x}}{\psi}, \cos\beta = \frac{\frac{\partial f}{\partial y}}{\psi}, \cos\gamma = \frac{\frac{\partial f}{\partial z}}{\psi}, \tag{4}$$

where
$$\psi = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}. \tag{5}$$

Substituting in (2) from (4), we get finally

$$A = \int_a^b \int_{\psi(x)}^{\phi(x)} \frac{\psi}{\frac{\partial f}{\partial z}}\, dy\, dx. \tag{6}$$

If the projection is made on either the XZ- or YZ-plane, the student can easily deduce the corresponding formula.

Ex. Find the area of the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

For convenience, we find the area of the part intercepted by the three coördinate planes, which is one eighth of the whole area. From the given equation we obtain at once

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 2z,$$

whence
$$\sec\gamma = \frac{2\sqrt{x^2 + y^2 + z^2}}{2z} = \frac{a}{z}.$$

From (6), therefore,
$$A = \int\int \frac{a}{z}\, dy\, dx.$$

The projection on the plane XY of the surface under consideration is the quadrant of the circle $x^2 + y^2 = a^2$, lying between the X- and Y- axes. Hence the limits of integration are respectively 0 and $\sqrt{a^2 - x^2}$ for y, and 0 and a for x. Inserting these limits and substituting for z, we get finally,

$$A = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{a\, dy\, dx}{\sqrt{a^2 - x^2 - y^2}}.$$

The first integration with x constant gives the result

$$a\int_0^{\sqrt{a^2-x^2}}\frac{dy}{\sqrt{a^2-x^2-y^2}} = a \text{ arc sin } \frac{y}{\sqrt{a^2-x^2}}\Bigg]_0^{\sqrt{a^2-x^2}} = a\,\frac{\pi}{2}\cdot$$

The second integration gives

$$A = \int_0^a \frac{\pi a}{2}\,dx = \frac{\pi a^2}{2}.$$

Hence the whole area is $8 \times \frac{\pi a^2}{2} = 4\,\pi a^2$.

EXERCISES.

1. Find by integration the area of the convex surface of the part of the cone

$$x^2 + y^2 = kz^2$$

between the planes $z = 0$ and $z = h$.

2. Find the surface of the cylinder $x^2 + z^2 = r^2$, included between the planes $y = 0$ and $y = mx$.

3. Two cylinders of equal base radius a intersect at right angles. Find the area of the surface of one intercepted by the other.

Suggestion: Take $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$ as the equations of the cylinders.

4. The reflector of an electric headlight is a paraboloid of revolution and has the dimensions shown in the figure. Find the area of the reflecting surface.

Suggestion: Project the surface on a plane containing the axis of the paraboloid.

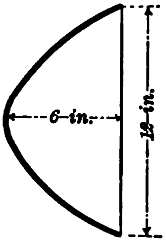

Fig. 99.

MISCELLANEOUS EXERCISES.

1. Find the following areas:

(a) Between the parabola $y^2 = 8\,x$ and the circle $x^2 + y^2 = 9$.

(b) Between the catenary $y = \cosh x$ [or $y = \frac{1}{2}(e^x + e^{-x})$] and the circle $x^2 + y^2 = 4$.

(c) The whole area of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

(d) The area of the loop of the curve $x^3 + y^3 = 3\,axy$.

2. By double integration, find the following areas:

(a) One loop of the curve $\rho = a \sin 2\,\theta$.

(b) The part of the curve $\rho = a \sin^3 \frac{3\,\theta}{3}$ below the X-axis.

(c) The parabola $\rho = a \sec^2 \frac{\theta}{2}$ between the vertex and the latus rectum.

3. Find the volume bounded by the cylinder $(x - a)^2 + (y - b)^2 = r^2$, the surface $xy = cz$, and the plane $z = 0$.

4. Find the volume of the solid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} = 1.$

5. Find the volume included between the surface $a^2x^2 + b^2z^2 = 2(ax + bz)y$ and the planes $y = \pm m$.

6. Find the volume generated by revolving the curve $\rho = a(1 + \cos\theta)$ about the initial line.

7. Find the area of that part of the surface of a sphere $x^2 + y^2 + z^2 = 2az$ lying within the paraboloid $z = mx^2 + ny^2$.

8. By extending the method of Art. 88, show that the length of a curve in space is given by the formula

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx.$$

9. From the formula of Ex. 8, find the length of the helix

$$z = a \operatorname{arc} \cos \frac{x}{b}, \quad z = a \operatorname{arc} \sin \frac{y}{b},$$

from $z = 0$ to $z = m$.

10. Find the area of the surface $z^2 + (x \cos\alpha + y \sin\alpha)^2 = a^2$ included between the three coördinate planes.

11. Find the area of the curved surface of a right cone whose base is the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, and whose altitude is c.

Suggestion: Taking the origin at the vertex of the cone, the equation of the surface is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \left(\frac{a}{c}z\right)^{\frac{2}{3}}$.

12. Find the volume of the pipe elbow, shown in Fig. 100.

Suggestion: Take the volume element as

$$\frac{\pi}{2}(R - \rho \sin\theta)\,\rho\, d\theta\, d\rho.$$

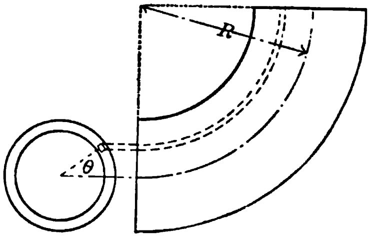

Fig. 100.

13. The axis of a right circular cylinder passes through the center of a sphere. The radius of the sphere is a, that of the cylinder is b, $(b < a)$. Find the volume of the part of the sphere external to the cylinder.

14. If $f(x, y)$ is a continuous point function, and ΔA is an element of a plane area, then

$$\underset{\Delta A \doteq 0}{L} \sum f(x, y)\,\Delta A = \int\int f(x, y)\, dy\, dx$$

is called the *surface integral* of the function $f(x, y)$ over the plane area in question, the integrals being taken between proper limits. (a) Find the surface integral of $f(x, y) = x^2 + y^2$ over a circle of radius a, origin at the center. (b) Find the surface integral of xy over that part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant.

15. Find the volume included between the surface $xy = cz$ and the planes $x = 0$, $x = b$, $y = 0$, $y = b$, $z = 0$, $z = a$.

CHAPTER XIV

PHYSICAL APPLICATIONS

118. Mass. Mean density. If m denote the mass and V the volume of a body, the ratio $\frac{m}{V}$ is called the *mean density* of the body. If we take a volume element ΔV, inclosing a point P, and denote the mass of the element by Δm, the ratio $\frac{\Delta m}{\Delta V}$ is the mean density of the element; and the limit of this ratio as ΔV approaches zero (still including the point P) is the density *at the point P*. If the density is the same at all points within a body, the body is said to be *homogeneous;* otherwise, it is said to be non-homogeneous.

Let the mass of a non-homogeneous body be divided into elements Δm_1, Δm_2, . . . Δm_n, whose volumes are respectively ΔV_1, ΔV_2, . . . ΔV_n. Denoting by γ_1, γ_2, . . . γ_n the mean densities of these elements, we have

$$\Delta m_1 = \gamma_1 \, \Delta V_1, \ \Delta m_2 = \gamma_2 \, \Delta V_2, \ . \ . \ . \ , \ \Delta m_n = \gamma_n \, \Delta V_n. \qquad (1)$$

The mass of the body is therefore

$$m = \underset{n=\infty}{L} \sum_1^n \Delta m_k = \underset{n=\infty}{L} \sum_1^n \gamma_k \, \Delta V_k = \int \gamma \, dV, \qquad (2)$$

and the mean density of the body, which we shall denote by $\bar{\gamma}$, is given by the equation

$$\bar{\gamma} = \frac{m}{V} = \int \frac{\gamma \, dV}{V}. \qquad (3)$$

If the mass is distributed continuously over a surface, we may replace the volume element ΔV by an area element ΔA; and if the mass is distributed along a curve, as, for example, along a thin wire, we replace ΔV by Δs, the element of length.

The volume element ΔV should be so chosen as to lead to the simplest integrations. Usually triple integration will be necessary, but in some cases the element may be taken in such a way that a double or even single integration is sufficient.

Ex. 1. Find the mean density of a sphere in which the density varies as the square of the distance from the center.

Polar coördinates are more convenient in this case. The density at a distance ρ from the center is $k\rho^2$, k being a constant; for the volume element ΔV we take $\rho^2 \sin\theta\, \Delta\rho\, \Delta\phi\, \Delta\theta$. From (3) we have therefore

$$\bar{\gamma} = \frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{a} k\rho^4 \sin\theta\, d\rho\, d\phi\, d\theta}{\frac{4}{3}\pi a^3} = \tfrac{3}{5}ka^2.$$

The nature of the problem is such that the result may also be obtained by a single integration. If we choose for the volume element a hemispherical shell of thickness $\Delta\rho$, we obtain $\Delta V = 4\pi\rho^2\Delta\rho$, whence

$$\bar{\gamma} = \frac{4\pi\int_0^{a} k\rho^4\, d\rho}{\frac{4}{3}\pi a^3} = \tfrac{3}{5}ka^2.$$

The mean density is therefore $\frac{3}{5}$ of the density at the surface.

Ex. 2. Find the mass and mean density of a thin plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, assuming that the density varies as the product xy.

In this case we choose the rectangular area element $\Delta A = \Delta x\, \Delta y$. The density may be denoted by kxy, k being a constant. Hence we have

$$m = \int_0^{a}\int_0^{\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}} kxy\, dy\, dx = \tfrac{1}{8}ka^2b^2;$$

$$\bar{\gamma} = \frac{\frac{1}{8}ka^2b^2}{\frac{1}{4}\pi ab} = \frac{1}{2\pi}kab.$$

EXERCISES.

1. Find the mass and mean density of a semicircular plate of radius a, whose density varies as the distance from the bounding diameter.

2. In Ex. 1 let the density vary as the distance from the center. Find the mass and mean density.

3. Find the mean density of a straight wire of length l, the density of which varies as the distance from one end.

4. Find the mass and mean density of a hemispherical solid, radius a, the density varying as the distance from the base.

5. Find the mass and mean density of a thin plate in the form of a right triangle in which the density varies as the distance from one of the short sides.

6. Given a right circular cone of height h, in which the density varies as the distance from a plane through the vertex perpendicular to the axis. Find the mean density.

119. First moments. Centroids. Let a given geometrical magnitude, line, surface, or solid, be divided in any convenient way into elements — a line into elements of length, a surface into area elements, a solid into volume elements. Let each element (Δs, ΔA, or ΔV) be multiplied by the distance of some chosen point within the element from a reference plane or line. The limit of the sum of these products as the elements are taken smaller and smaller is called the *first moment* of the geometrical magnitude with respect to the given plane or line. We shall denote first moments by the symbol M with proper subscripts.

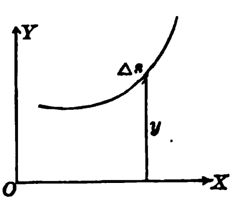

Fig. 101.

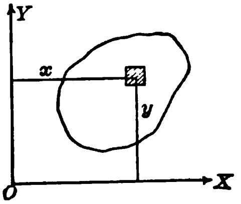

Fig. 102.

From the definition we have for the first moment M_x of a plane curve with respect to the X-axis, Fig. 101,

$$M_x = \underset{\Delta s \doteq 0}{L} \sum y \; \Delta s = \int y \, ds; \tag{1}$$

and for the first moment of a plane area with respect to the same axis, Fig. 102,

$$M_x = \underset{\Delta A \doteq 0}{L} \sum y \; \Delta A = \int y \, dA. \tag{2}$$

The first moment of a solid with respect to one of the coördinate planes, say the XY-plane (see Fig. 93), is given by the equation

$$M_{xy} = \underset{\Delta V \doteq 0}{L} \sum z \; \Delta V = \int z \, dV. \tag{3}$$

For ΔA and ΔV proper area or volume elements are to be substituted before integration is attempted. In some cases the elements may be so chosen that a single integration is sufficient, but in general double or triple integration will be required.

The first moment of the mass of a solid is derived from the corresponding moments of the geometrical magnitude by the introduction of a density factor. Thus, denoting the density by γ, the first moment of a mass distributed throughout a given region with reference to the XY-plane is

$$M_{xy} = \int \gamma\, z dV,$$

the limits of integration being taken so as to include the region in question.

Let a given solid be referred to a system of rectangular coördinates, and let M_{xy}, M_{yz}, M_{zx} denote the first moments with respect to the three coördinate planes. It is possible to find a point G whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$, are given by the equations,

$$\left.\begin{aligned} \bar{x} &= \frac{M_{yz}}{V} = \frac{\int x\, dV}{V} \\ \bar{y} &= \frac{M_{zx}}{V} = \frac{\int y\, dV}{V} \\ \bar{z} &= \frac{M_{xy}}{V} = \frac{\int z\, dV}{V} \end{aligned}\right\} \tag{4}$$

The point G thus defined is called the *centroid* of the volume V.

If we replace V and dV by A and dA, formulas (4) give the centroid of a plane surface. Likewise the centroid of a curve may be obtained by using s and ds, where s denotes the length of the curve.

If in (4) we substitute the mass m of the solid for its volume V, the resulting values of $\bar{x}$, $\bar{y}$, and $\bar{z}$ are the coördinates of the centroid of the mass.* If the solid is homogeneous, the constant density factor γ, which appears in each member of the equation in the first degree, may be dropped, and in this case the centroid of the mass and that of the volume coincide. If, however, the solid is not homogeneous, γ is variable, and we have

$$m = \underset{\Delta V \doteq 0}{L} \sum \gamma\, \Delta V = \int \gamma\, dV,$$

$$M_{yz} = \int \gamma x\, dV,$$

whence

$$\bar{x} = \frac{\int \gamma x\, dV}{\int \gamma\, dV}, \quad \bar{y} = \frac{\int \gamma y\, dV}{\int \gamma\, dV}, \quad \bar{z} = \frac{\int \gamma z\, dV}{\int \gamma\, dV}. \tag{5}$$

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* The centroid of a mass is often called the *center of gravity* of the mass.

120. General theorems relating to centroids. The following general theorems are useful in the determination of centroids.

THEOREM I. *The first moment of a volume (or mass) with respect to a plane or axis containing the centroid is zero.*

If the centroidal plane (i.e., plane containing the centroid) is a coördinate plane, the theorem follows at once from (4) or (5). Thus, if the centroid lies in the YZ-plane, $\bar{x} = 0$, whence $M_{yz} = 0$. It can be readily shown that the theorem holds for *any* centroidal plane; therefore it holds for the intersection of two such planes, that is, a centroidal axis.

THEOREM II. *A plane of symmetry of a solid or of a homogeneous mass is a centroidal plane. Likewise an axis of symmetry of a plane figure is a centroidal axis.*

Take the plane of symmetry as a coördinate plane, say the XY-plane. To an element ΔV above the plane there corresponds an equal element at the same distance below the plane. The moment of the first element with respect to the plane is $z_1 \Delta V$, that of the second element is $- z_1 \Delta V$, whence the moment of the pair of elements is zero. Since all the elements of the solid can be thus arranged in pairs, the first moment of the solid with respect to the plane is zero.

THEOREM III. *If $V_1, V_2, \ldots V_n$ are n volumes and if $\bar{x}_1, \bar{x}_2, \bar{x}_3 \ldots \bar{x}_n$ are the x-coördinates of their centroids, then the x-coördinate of the centroid of the system of volumes is*

$$\bar{x} = \frac{V_1\bar{x}_1 + V_2\bar{x}_2 + \ldots + V_n\bar{x}_n}{V_1 + V_2 + \ldots + V_n}. \tag{1}$$

A similar theorem holds for n masses, $m_1, m_2, \ldots m_n$, or for n plane areas in the same plane. The numerator of the second member of (1) is the moment M_{yz} of the system with respect to the YZ-plane, and the denominator is the total volume; hence the theorem follows from (4), Art. 119.

THEOREM IV. (*Theorems of Pappus and Guldin.*)

(a) *If a plane curve is revolved about an axis in its plane, the area of the surface generated is equal to the product of the length of the curve and the circumference of the circle described by the centroid of the curve.*

(b) *If a plane figure is revolved about an axis in its plane, the volume of the solid of revolution generated is equal to the product of the area of the figure and the circumference of the circle described by the centroid of the area.*

From (4), Art. 119, we have for a plane curve

$$\bar{y} = \frac{\int y\,ds}{s}, \quad \text{or} \quad s\bar{y} = \int y\,ds,$$

where s denotes the length of the curve. The surface generated by the revolution of the curve about the X-axis is $S = 2\pi \int y\,ds$ (see Art. 90). Hence equating the two expressions for $\int y\,ds$, we obtain

$$S = 2\,\pi\,\bar{y}\,s. \tag{2}$$

Evidently the theorem holds equally well for an incomplete revolution.

To prove the second theorem, let the axis of revolution be taken as the axis of x, as before; then denoting by ΔA an element of the plane area, we have for the volume generated by a complete revolution of the area,

$$V = \underset{\Delta A \doteq 0}{L} \sum 2\,\pi y\,\Delta A = 2\,\pi \int y\,dA.$$

But

$$\int y\,dA = A\,\bar{y};$$

hence

$$V = 2\,\pi\,\bar{y}A. \tag{3}$$

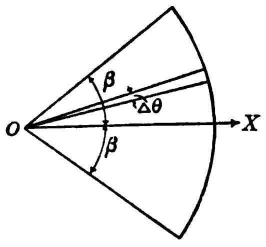

Fig. 103.

Ex. 1. Find the coördinates of the centroid of a circular arc of radius a which subtends an angle $2\,\beta$ at the center. Fig. 103.

Take the line of symmetry as the X-axis; then $\bar{y} = 0$, and the centroid lies in this line. Polar coördinates lead to the simplest integral; hence taking $x = a \cos\theta$, $\Delta s = a\,\Delta\theta$, we get,

$$\bar{x} = \frac{M_y}{s} = 2\int_0^{\beta} \frac{a^2 \cos\theta\,d\theta}{2\,a\beta} = \frac{2\,a^2 \sin\beta}{2\,a\beta} = \frac{a\sin\beta}{\beta}.$$

For a semi-circumference $\beta = \frac{\pi}{2}$, whence $\bar{x} = \frac{2\,a}{\pi}$.

Ex. 2. Find the coördinates of the centroid of the plate described in illustrative example 2, Art. 119.

From the defining equation $\bar{x} = \dfrac{\int \gamma\, x\, dA}{\int \gamma\, dA}$ we have upon substituting kxy for γ and $dy\, dx$ for dA,

$$\bar{x} = \frac{\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} k\, x^2 y\, dy\, dx}{\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} kxy\, dy\, dx} = \tfrac{8}{15} a.$$

Similarly $$\bar{y} = \tfrac{8}{15} b.$$

Ex. 3. Find the centroid of a semicircle of radius a.

Taking the X-axis as the axis of symmetry, the length of the path described by the centroid as the semicircle is revolved about the Y-axis is $2\pi\bar{x}$. The semicircle by its revolution describes a sphere whose volume is $\frac{4}{3}\pi a^3$; hence by the second theorem of Pappus,

$$2\pi\bar{x} \cdot \tfrac{1}{2}\pi a^2 = \tfrac{4}{3}\pi a^3,$$

or $$\bar{x} = \frac{4a}{3\pi}.$$

Ex. 4. Find the centroid of a wedge-shaped solid, Fig. 104, cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$, $\dfrac{z}{b} + \dfrac{x}{a} = 1$.

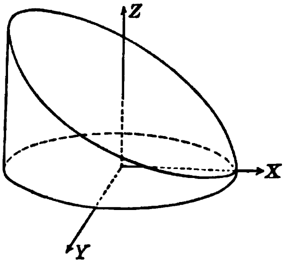

Fig. 104.

The volume of the solid is $\frac{1}{2}\pi a^2 \cdot 2b = a^2 b$. The first moment with respect to the XY-plane is given by the integral $\int\int\int z\, dz\, dy\, dx$. The limits of integration, as determined by an inspection of the solid, are as follows: For z, 0 and $b\left(1 - \dfrac{x}{a}\right)$; for y, $-\sqrt{a^2 - x^2}$, $\sqrt{a^2 - x^2}$; for x, $-a$ and a. Hence we have

$$M_{xy} = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{b\left(1-\frac{x}{a}\right)} z\, dz\, dy\, dx$$

$$= b^2 \int_{-a}^{a} \left(1 - \frac{x}{a}\right)^2 \sqrt{a^2 - x^2}\, dx = \tfrac{5}{8}\pi a^2 b^2.$$

For the moment with respect to the YZ-plane, we have

$$M_{zy} = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{b\left(1-\frac{x}{a}\right)} x\, dz\, dy\, dx$$

$$= 2b \int_{-a}^{a} x\left(1 - \frac{x}{a}\right) \sqrt{a^2 - x^2}\, dx = -\frac{\pi a^3 b}{4}.$$

Since the XZ-plane is a plane of symmetry, $\overline{y} = 0$. The coördinates of the centroid are therefore

$$\overline{x} = -\frac{\frac{1}{4}\pi a^3 b}{\pi a^2 b} = -\tfrac{1}{4}a, \quad \overline{y} = 0, \quad \overline{z} = \frac{\frac{5}{8}\pi a^2 b^2}{\pi a^2 b} = \tfrac{5}{8}b.$$

EXERCISES.

1. Find the coördinates of the centroids of the following plane curves:

(a) A semi-circumference.

(b) An arc of the parabola $y^2 = 4ax$ from $x = 0$ to $x = a$.

(c) An arc of the catenary $y = a \cosh \frac{x}{a}$ from $x = 0$ to $x = a$.

(d) The cycloid $x = a\theta - a \sin \theta$, $y = a - a \cos \theta$.

2. Find the coördinates of the centroid of the plane area between the curve in Ex. 1 and the X-axis.

3. Find the centroid of the area between the cissoid $y^2 = \dfrac{x^3}{a - x}$ and its asymptote $x = a$.

4. Find the centroid of the area between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the axes.

5. Find the centroid of a segment of the parabola $y^2 = 4ax$ cut off by the double ordinate corresponding to $x = a$, assuming that the density at any point is proportional to the distance from the Y-axis.

6. Find the centroid of a semicircle of radius a, assuming the density proportional to the distance from the bounding diameter.

7. Find the centroid of a segment of a homogeneous paraboloid of revolution between the origin and $x = a$.

8. Find the centroid of a hemisphere whose density varies as the distance from the base.

9. Find the centroid of a surface generated by the revolution of one quadrant of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

10. Find the coördinates of the centroid of one loop of the curve

$$\rho = a \sin 2\,\theta.$$

11. Find the coördinates of the centroid of the mass of one eighth of a sphere included between the coördinate planes, assuming that the density varies as the distance from the origin.

12. From Theorem III, Art. 120, show that if points P and Q are the centroids of two volumes V_1 and V_2 (or areas A_1 and A_2, or masses m_1 and m_2), the centroid of the whole system $V_1 + V_2$ lies on the line joining P and Q and divides it into segments inversely proportional to V_1 and V_2.

13. A solid is composed of a right circular cylinder and a right cone. See Fig. 105. Find the centroid of the solid.

14. Three masses, $m_1 = 3$, $m_2 = 4$, and $m_3 = 5$ have their centroids located at the three vertices of an equilateral triangle the side of which has the length a. By means of Theorem III, find the position of the centroid of the system.

15. By the Theorem of Pappus find the centroid of the semi-circumference of a circle of radius a.

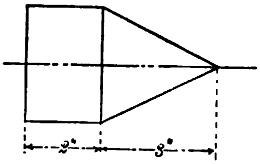

Fig. 105.

121. Second moment. Radius of gyration. Let each of the elements into which a volume (or area, length, or mass) is divided be multiplied by the *square* of the distance of some chosen point from a reference line or plane. The limit of the sum of these products as the elements are taken smaller is called the *second moment* * of the magnitude in question with respect to the line or plane of reference.

Formulas for second moments may therefore be derived from those for first moments by squaring the distance factor. Denoting second moment by the general symbol I, we have the following formulas corresponding to (1), (2), and (3) of Art. 119.

For a plane curve, $$I_x = \int y^2\,ds. \qquad (1)$$

For a plane area, $$I_x = \int y^2\,dA. \qquad (2)$$

For a volume, $$I_{xy} = \int z^2\,dV. \qquad (3)$$

From its definition, the second moment of a volume must have for its physical dimensions,

$$\text{volume} \times (\text{length})^2;$$

hence the second moment of a volume V can be expressed in the form,

$$I = Vk^2 \qquad (4)$$

in which k is a length. Similarly the second moment of an area A may be expressed in the form

$$I = Ak^2, \qquad (5)$$

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* The second moment of a mass is sometimes called the *moment of inertia* of the mass.

and likewise for any magnitude whose second moment is taken. The length k as defined by (4) or (5) is called the *radius of gyration* of the given magnitude with respect to the reference plane or axis.

Ex. 1. Find the second moment and the radius of gyration of the area of a semicircle of radius a with respect to the bounding diameter.

Taking the origin at the center, and the bounding diameter as the axis of x, we have by transforming to polar coördinates $y = \rho \sin \theta$, $\Delta A = \rho\, \Delta\theta\, \Delta\rho$. Therefore

$$I_x = \int_0^a \int_0^\pi \rho^2 \sin^2 \theta \cdot \rho\, d\theta\, d\rho = \tfrac{1}{8} \pi a^4$$

and

$$k^2 = \frac{I_x}{A} = \frac{\frac{1}{8} \pi a^4}{\frac{1}{2} \pi a^2} = \frac{1}{4} a^2,$$

whence

$$k = \tfrac{1}{2} a.$$

Ex. 2. Find the second moment of a straight rod or wire of length l about one end, assuming that the density varies as the distance from that end.

Consider the rod as lying in the X-axis with the end about which the moment is taken at the origin. At a distance x from the origin the density is kx and the mass of an element of length Δx is $kx\, \Delta x$. The second moment of this mass element about the origin is $x^2 \cdot kx\, \Delta x$; hence the second moment of the rod is

$$I = \underset{\Delta x \doteq 0}{L} \sum x^2 \cdot kx\, \Delta x = \int_0^l kx^3\, dx = \tfrac{1}{4} kl^4.$$

The mass of the rod is

$$m = \int_0^l kx\, dx = \tfrac{1}{2} kl^2.$$

Therefore

$$k^2 = \frac{I}{m} = \frac{1}{2} l^2.$$

EXERCISES.

Find the second moment and the square of the radius of gyration in the following cases.

1. A rectangle of width b and length h about an axis coinciding with the short side b.

2. The same rectangle about an axis through the center parallel to the side b.

3. A circle of radius a about an axis through the center perpendicular to the plane of the circle.

4. The same circle as in Ex. 3 with the density varying as the distance from the center.

5. Triangular area of base b and height h about an axis through the vertex parallel to the base.

6. A semi-circumference of a circle of radius a about the diameter.

7. The area between the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and the X-axis about the X-axis.

8. The cycloidal curve about the X-axis.

9. Solid sphere of radius a with respect to a diametral plane.

10. The same sphere as in Ex. 9, assuming that the density varies as the distance from the diametral plane.

122. General theorems relating to second moments. The following general theorems are useful in the determination of second moments and radii of gyration.

THEOREM I. *If a solid (or homogeneous mass) is divided by a plane of symmetry, the second moment of the entire solid (or mass) with respect to this plane is double that of the part on one side of the plane of symmetry.*

This theorem is almost self-evident. For any element ΔV on one side of the plane at a distance y there is a corresponding equal element on the other side at a distance $-y$. The second moment of one element is $y^2\Delta V$, that of both is $y^2\Delta V + (-y)^2\Delta V = 2y^2\Delta V$. Since all the elements of the solid can thus be arranged in pairs, the theorem follows at once. The theorem holds also for the second moment of a plane surface with respect to an axis of symmetry.

THEOREM II. *The second moment of a solid (or mass) with respect to a line is the sum of the second moments with respect to two planes which intersect at right angles in the line.*

Consider the given line as the X-axis, and take the XZ- and XY-planes as the intersecting planes. A point P in a chosen volume element has the coördinates (x, y, z). Its distance from the X-axis is $\sqrt{y^2 + z^2}$, and by the definition the second moment of the solid with respect to this axis is

$$I_x = \int (y^2 + z^2)\, dV.$$

But $$\int y^2\, dV = I_{xz}, \text{ and } \int z^2\, dV = I_{xy};$$

hence $$I_x = I_{xz} + I_{xy}. \tag{1}$$

A similar theorem applies to plane areas. Referring to Fig. 102, it is clear that the sum of I_x and I_y, the second moments of the plane area with respect to the axes of x and y, respectively, is the second moment of the area with respect to an axis through the origin O perpendicular to the plane of the figure.

THEOREM III. *The second moment of a solid (or mass) with respect to an axis is equal to its second moment with respect to a parallel axis through the centroid plus the product of the volume (or mass) of the solid and the square of the distance between the axes.*

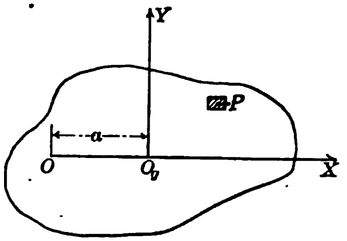

Fig. 106.

Let the given axis pierce the plane of the page at O, Fig. 106, and suppose O_g to be the trace of the parallel axis through the centroid. Take the line OO_g as the X-axis, and let a denote the distance OO_g. Referred to O_g as an origin, the point P in a given volume element has the coördinates (x, y); referred to O as origin, the coördinates are $(x + a, y)$. The second moment of the solid with respect to the axis through O_g is therefore,

$$I_g = \int (x^2 + y^2)\, dV,$$

while the second moment with respect to the axis through O is

$$\begin{aligned} I &= \int [(x + a)^2 + y^2]\, dV \\ &= \int (x^2 + y^2)\, dV + a^2 \int dV + 2\, a \int x\, dV \\ &= I_g + a^2 V + 2\, a \int x\, dV. \end{aligned}$$

Now $\int x\, dV$ is the first moment of the solid about the axis through O_g; hence since this axis contains the centroid, $\int x\, dV = 0$, and we have finally,

$$I = I_g + a^2 V. \tag{2}$$

Dividing both members of (2) by V, we get

$$k^2 = k_g^2 + a^2. \tag{3}$$

The student may show that the theorem holds for parallel planes, one of which contains the centroid.

THEOREM IV. *If $k_1, k_2, \ldots k_n$ are the radii of gyration of n volumes (or areas or masses) with respect to a line or plane the radius of gyration k of the entire system with respect to the line or plane is given by*

$$k^2 = \frac{v_1k_1^2 + v_2k_2^2 + \ldots + v_nk_n^2}{v_1 + v_2 + \ldots + v_n}.$$

This theorem is useful in finding the second moment of an area or volume which can be conveniently divided into parts. The proof is similar to that of Theorem III, Art. 120.

Ex. 1. Find the radius of gyration of a circle of radius a about a tangent.

From Theorem I, the radius of gyration about a diameter is the same as that for a semicircle; hence from Ex. 1, Art. 121, $k_g^2 = \frac{1}{4}a^2$. Using Theorem III, we obtain therefore,

$$k^2 = \tfrac{1}{4}a^2 + a^2 = \tfrac{5}{4}a^2.$$

Ex. 2. Find the second moment of a right circular cone with respect to a line through the vertex perpendicular to the axis of the solid. Fig. 107.

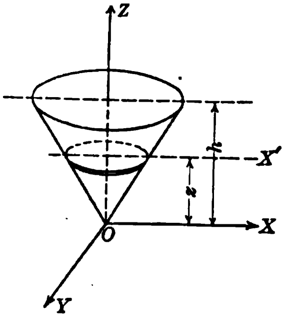

Fig. 107.

Take the axis of the cone as the Z-axis, and choose the vertex as the origin. Let h be the altitude and b the base radius. Consider a slice of the cone parallel to the XY-plane at a distance z from it. The radius of the slice is $\frac{zb}{h}$, its thickness is Δz, and its second moment with respect to a diametral axis X', parallel to OX, is approximately

$$\frac{1}{4}\pi\frac{b^4}{h^4}z^4\,\Delta z.$$

The second moment of this slice with respect to OX is therefore

$$\frac{1}{4}\pi\frac{b^4}{h^4}z^4\,\Delta z + \pi\frac{b^2}{h^2}z^2\,\Delta z\cdot z^2 = \pi\frac{b^2}{h^2}\left(\frac{1}{4}\frac{b^2}{h^2} + 1\right)z^4\,\Delta z.$$

The second moment of the cone about OX is therefore the limit of the sum of terms of this type; that is,

$$I = \pi\frac{b^2}{h^2}\left(\frac{b^2}{4h^2} + 1\right)\int_0^h z^4\,dz = \left(\frac{b^2}{4h^2} + 1\right)\frac{\pi b^2h^3}{5}.$$

The radius of gyration is given by the equation

$$k^2 = \frac{3h^2}{5}\left(\frac{b^2}{4h^2} + 1\right) = \frac{3}{20}\left(b^2 + 4h^2\right).$$

EXERCISES.

Determine the radii of gyration of the following:

1. A square whose side is a; (a) about an axis through the center perpendicular to its plane; (b) about a diagonal.

Suggestion: Make use of Theorem II.

2. An ellipse with semi-axes a and b; (a) for the major axis; (b) for the minor axis; (c) for a tangent at the end of the minor axis; (d) for a centroidal axis perpendicular to the plane of the ellipse.

3. A ring bounded by circles of radii a_1 and a_2, respectively, about a diameter. Use Theorem IV.

4. A sphere of radius a about a diameter.

5. A right circular cone; (a) about a plane containing the axis; (b) about the axis. Take a as base radius and h as altitude.

6. A right circular cylinder; (a) about its axis; (b) about a generating element; (c) about a diameter of one base.

7. Show that the radius of gyration of a right circular cylinder about its axis is the same as that of the circular cross-section about the same axis. Deduce a similar principle for prismatic solids in general.

8. Find the position of the axis x_c passing through the centroid of the cross-section, shown in Fig. 108.

9. Determine the radius of gyration of the cross-section, Fig. 108:

(a) with respect to the axis OX;

(b) with respect to the axis x_c.

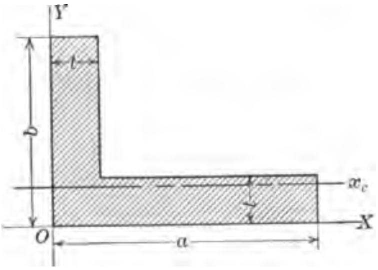

Fig. 108.

123. Stress. Liquid pressure. The term *stress* is used to designate a system of forces, usually parallel, distributed over a surface. Examples are found in the pressure between bodies in contact, and in particular in the pressure of liquids upon containing surfaces. We shall consider only *normal* stresses, that is, stresses in which the action lines of the parallel forces are perpendicular to the surface. All surfaces are assumed to be plane surfaces.

The intensity of stress at a given point of the surface is the limit of the ratio $\frac{\Delta F}{\Delta A}$, where ΔF denotes the stress over an area of element

ΔA which includes the point. Denoting by p the intensity at the point, we have therefore

$$p = \underset{\Delta A \to 0}{L} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}. \tag{1}$$

To determine the mean intensity $\bar{p}$ of the pressure on a given plane surface, we divide the surface into elements ΔA_1, ΔA_2, etc., and multiply each by the intensity at some point within the element. The limit of the sum of these products, that is,

$$\underset{\Delta A \to 0}{L} \sum p \, \Delta A = \int p \, dA, \tag{2}$$

is the total stress on the surface, and this divided by the total area gives the mean intensity. Hence we have

$$\bar{p} = \frac{\int p dA}{A}. \tag{3}$$

The area element ΔA should be chosen so as to lead to the simplest integration.

As an example, let us consider the pressure of a liquid on a plane surface. It is a well-known law of hydrostatics that the intensity of pressure at any point of the surface is proportional to the depth of the point below the liquid surface. Thus if XX, Fig. 109, is the free surface, the intensity at the point P at a distance y below XX is $p_0 y$, where p_0 denotes the intensity at a unit distance below XX. The mean intensity of pressure on the surface is

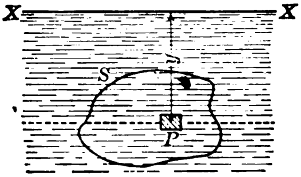

Fig. 109.

$$\bar{p} = \int \frac{p \, dA}{A} = p_0 \frac{\int y \, dA}{A} = p_0 \bar{y}, \tag{4}$$

where $\bar{y}$ is the distance of the centroid of the surface below XX. Hence the mean intensity of liquid pressure on a submerged plane surface is equal to the intensity at the centroid of the surface.

The point in which the action line of the resultant of the parallel forces that constitute the stress pierces the plane surface is called the *center of stress.* In case the stress is a liquid pressure, the point is called the *center of pressure.* The position of this point in the plane surface is determined by the principle of statics that the sum of the moments of several parallel forces about an axis is equal to the moment of their resultant about the same axis.

In the case of the liquid pressure on a plane area, let the line XX, in which the plane of the area cuts the liquid surface, be the axis of moments, and let y_c denote the distance of the center of pressure from this axis. The product $p\,\Delta A$ is the pressure on an area element ΔA, and the moment of this pressure about XX is $yp\,\Delta A$; hence the limit $\underset{\Delta A \doteq 0}{L} \sum yp\,\Delta A$ gives the moment of the liquid pressure about XX. The resultant of all the forces $p\,\Delta A$, that is, the total pressure on the area, is $\underset{\Delta A \doteq 0}{L} \sum p\,\Delta A$, and the moment of this resultant about XX is $y_c \underset{\Delta A \doteq 0}{L} \sum p\,\Delta A$. Therefore we have

$$y_c \underset{\Delta A \doteq 0}{L} \sum p\,\Delta A = \underset{\Delta A \doteq 0}{L} \sum yp\,\Delta A,$$

whence
$$y_c = \frac{\int yp\,dA}{\int p\,dA}. \tag{5}$$

By choosing a Y-axis perpendicular to XX we can in a similar manner derive the formula

$$x_c = \frac{\int xp\,dA}{\int p\,dA}. \tag{6}$$

Formulas (5) and (6), while derived for liquid pressure, hold good for any normal stress on a plane area. In the case of liquid pressure, (5) becomes upon the substitution of $p_0 y$ for p,

$$y_c = \frac{\int y^2\,dA}{\int y\,dA} = \frac{Ak^2}{A\bar{y}} = \frac{k^2}{\bar{y}}, \tag{7}$$

in which k is the radius of gyration of the area with respect to XX, and $\bar{y}$ is the coördinate of the centroid.

EXERCISES.

1. The stress on a rectangular area of width b and height h, varies as the square of the distance from the horizontal base. Find the center of stress, and derive an expression for the total stress on the area.

2. Same as Ex. 1, except that the stress varies as the square root of the distance from the base.

3. Find the center of pressure of a submerged rectangular disk of breadth b with edges horizontal. Let h_1 and h_2 be the distances of the upper and lower edges from the surface.

4. Find the center of pressure of a triangle, having its vertex in the surface and the opposite side horizontal. Denote base by b, altitude by h.

5. Find the center of pressure of the same triangle if the vertex is at a distance c below the surface.

6. Find the center of pressure of a submerged semicircular disk of radius a, with its diameter in the free surface of the liquid.

124. Discharge through orifices. It is a well-known law of hydromechanics that the velocity v with which a liquid flows from a small orifice in a vessel is given theoretically by the equation

$$v = \sqrt{2\,gh}, \tag{1}$$

where h is the head or height of the liquid surface above the orifice. If then A denotes the area of the orifice, and Q the volume flowing, we have for the discharge under ideal conditions

$$Q = Av = A\sqrt{2\,gh}. \tag{2}$$

Because of contraction of the jet and other causes, the discharge determined experimentally is smaller than that calculated by (2).

In the case of a large orifice, as shown in Fig. 110, the heads at different parts of the orifice are not equal, and consequently the velocity is not the same at all points. We may assume, however, that the flow across an element ΔA of the orifice is as it would be through a small orifice of the same dimensions in the wall. Let $h_1, h_2, \ldots, h_n$, etc., denote the heads at chosen points within the elements ΔA_1, ΔA_2, etc., respectively. Then the total discharge is the limit of the sum

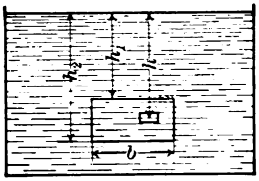

Fig. 110.

$$\Delta A_1\sqrt{2\,gh_1} + \Delta A_2\sqrt{2\,gh_2} + \ldots + \Delta A_n\sqrt{2\,gh_n};$$

that is, $$Q = \underset{\Delta A \doteq 0}{L} \sum \sqrt{2\,gh}\,\Delta A = \int \sqrt{2\,gh}\,dA. \tag{3}$$

Ex. Find the ideal discharge through a rectangular orifice of width b, Fig. 110.

The area element may be taken as a horizontal strip of length b and width Δh; hence $\Delta A = b\Delta h$, and we have therefore

$$Q = \int_{h_1}^{h_2} \sqrt{2\,gh}\,b\,dh = \tfrac{2}{3}\,b\sqrt{2\,g}\,(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}).$$

EXERCISES.

1. Derive an expression for the discharge through the triangular orifice, Fig. 111.

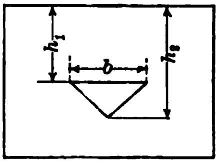

Fig. 111.

2. Find the discharge through an opening in the form of a trapezoid with the parallel sides b_1 and b_2 horizontal and at distances h_1 and h_2 respectively below the liquid surface.

3. Find the discharge through a triangular notch, that is, a triangular orifice with its base in the liquid surface.

MISCELLANEOUS EXERCISES.

1. Find the mean density of a right circular cone whose density varies as the distance from the base.

2. Find the mean density and mass of a solid hemisphere of radius a, assuming that the density varies as the distance from a tangent plane parallel to the base.

3. Using the theorems of Pappus, find the volume and surface of a ring formed by revolving a circle of radius a about an axis at a distance b from the center of the circle, $b > a$.

4. A helical screw thread whose cross section is an isosceles triangle is cut on a cylinder. The radius of the cylinder to the root of the thread is r, and the height of the thread is a. Find by the theorem of Pappus the volume of one turn of the thread.

5. Apply the theorem of Pappus to find the surface and volume of a right circular cone.

6. Using the theorem of Pappus, show that an element of the volume of a surface of revolution is $2\,\pi\rho^2 \sin\phi\,\Delta\rho\,\Delta\phi$, and that the volume is therefore given by the formula

$$V = 2\,\pi \int\int \rho^2 \sin\phi\,d\rho\,d\phi,$$

with proper limits of integration.

7. Using the theorem of Pappus, deduce the general form for the polar volume element, viz.: $\Delta V = \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$.

8. A fly-wheel rim has a section shown in Fig. 112; the outer circumference is 5 feet from the shaft axis.

(a) Find the radius of gyration of the rim with respect to the shaft axis.

(b) Find the radius of gyration of a plane section of the rim about the same axis.

Explain why the two results are different.

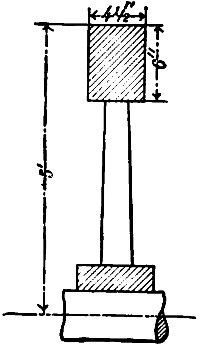

Fig. 112.

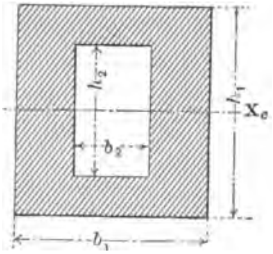

Fig. 113.

9. Find the radius of gyration of the hollow rectangle, Fig. 113, about the axis X_c, which passes through the center of the figure.

10. In Fig. 114, the thickness of the plates a, a and channels b, b is ⅜ inch throughout. Find the length e in order that the radius of gyration with respect to the axis YY shall be the same as that with respect to the axis XX.

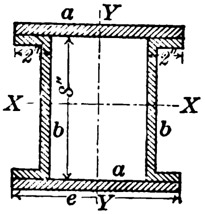

Fig. 114.

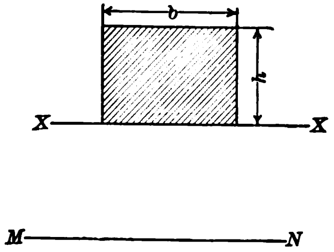

Fig. 115.

11. The intensity of the force acting at any point in the rectangle, Fig. 115, is proportional to the distance of the point from the axis XX along one side, and inversely proportional to the distance from a parallel axis MN at a distance c from XX. Find the mean intensity of the force and the total force acting on the rectangle.

12. A pipe elbow, Fig. 116, is to be provided with a foot a to support it in the position shown. The vertical line MM through the foot should therefore pass through the centroid of the elbow. Find the distance c.

13. The following graphical method is used for finding the radius of gyration of the cross section of a hollow cylindrical column.* *Lay off to a scale of* 1 *inch equal to* 4 (*or* 40) *the inner and outer radii as the legs of a right triangle; then the hypotenuse measured to a scale of* 1 *inch equal to* 1 (*or* 10) *is the radius of gyration sought.* Give a proof of this construction.

14. From particular examples obtain the following mnemonic rule due to Routh: *For homogeneous masses with axes of symmetry, the square of the radius of gyration is* ⅓, ¼, *or* ⅕ *of the sum of the squares of the perpendicular semi-axes, according as the mass is rectangular, elliptic, or ellipsoidal.*

15. From the second moment of a sphere about a diameter deduce by differentiation the second moment of a spherical shell about a diameter.

16. Find the centroid of the volume $OABCD$, Fig. 117, formed by the three coördinate planes and the warped surface generated by the line EF, which remaining parallel to the XY-plane slides on the lines AD and BC.

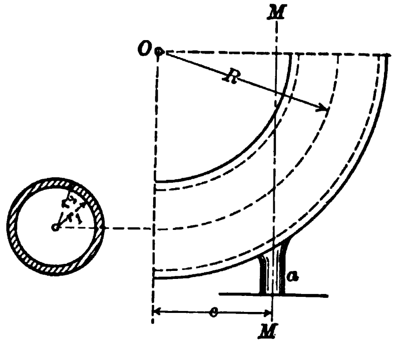

Fig. 116.

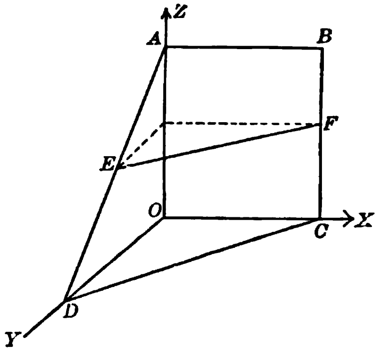

Fig. 117.

17. Following the methods of Art. 117, show that the centroid of a surface $f(x, y, z) = 0$ is given by the equations

$$\bar{x} = \frac{\displaystyle\int\!\!\int \frac{\phi}{\frac{\partial f}{\partial x}}\, x\, dz\, dy}{\displaystyle\int\!\!\int \frac{\phi}{\frac{\partial f}{\partial x}}\, dz\, dy}, \text{ etc.}$$

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* B. F. LaRue in *Engineering News*, Feb. 2, 1893.

18. (a) For a surface of revolution show that the radius of gyration about the axis of revolution is given by the equation

$$k^2 = \frac{\int y^3\,ds}{\int y\,ds},$$

where the X-axis is taken as the axis of revolution. (b) Show that for the corresponding solid of revolution

$$k^2 = \frac{\int y^4\,dx}{\int y^2\,dx}.$$

CHAPTER XV

INFINITE SERIES

125. Fundamental definitions. From his study of algebra, the student is already familiar with the elementary properties of infinite series. In the present chapter, we shall recall briefly some of the more important of those properties and develop still others.

Let $u_1, u_2, u_3, \ldots$ be an infinite succession of values following one another according to some fixed law. We denote the sum of the first n of these values by S_n, that is,

$$S_n = u_1 + u_2 + \ldots + u_n. \tag{1}$$

When n becomes infinite, we have the infinite series

$$u_1 + u_2 + u_3 + \ldots \tag{2}$$

If S_n has a limit as n becomes infinite, that limit is called the *sum* of an infinite series. The series (2) is said to be *convergent* if we have

$$\underset{n=\infty}{L} S_n = A, \tag{3}$$

where A is a definite number; in all other cases the series is said to be *divergent*. As n increases indefinitely, S_n may fail to have a limit, and therefore the series be divergent either because S_n oscillates between two numbers, as in the series

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots,$$

or because S_n ultimately exceeds every finite number.

The terms of the series may be functions of some variable x. Then the series is said to converge for any particular value of this variable, say $x = x_0$, when, if x is replaced by x_0 in each term, we have

$$\underset{n=\infty}{L} S_n(x_0) = A.$$

If the corresponding limits exist for all values of x in a certain interval (α, β), the series is said to converge throughout the interval and to define a function in the interval. We may then write,

$$f(x) = \underset{n=\infty}{L} S_n(x), \qquad \alpha \leqq x \leqq \beta.$$

Ex. 1. Let the given series be

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

We have then

$$\underset{n=\infty}{L}\ S_n(x) = \underset{n=\infty}{L}\ \frac{1 - x^n}{1 - x}.$$

This limit is $\frac{1}{1-x}$ for all values of x within the interval $-1 < x < +1$. Hence within this interval the series defines the function

$$f(x) = \frac{1}{1-x}.$$

It is to be noted that in this case $f(x)$ is defined by the series only for the interval $(-1, 1)$, the end points being excluded. For $|x| \geqq 1$ the series becomes divergent and does not define a function.

Ex. 2. Consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

This series may be written in the form

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots$$
$$+ \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right) + \ldots$$

The sum of the terms in each of these parentheses is greater than $\frac{1}{2}$, and as the number of such sums that can be formed from the given series is indefinitely large, we have

$$\underset{n=\infty}{L}\ S_n > \underset{n=\infty}{L}\ (n \cdot \tfrac{1}{2}) = \infty.$$

This result leads to an important observation; namely, that a series is not necessarily convergent because the terms themselves decrease as n increases.

If a series containing negative terms is still convergent when all of the negative terms are taken positively, that is, when only the absolute values of the terms are considered, the series is said to converge *absolutely* or *unconditionally.* If the series is convergent when the negative terms are taken with their proper signs, but is not convergent when those signs are taken positively, the series is said to converge *conditionally.* The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$$

is such a series. On the other hand, the series

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$$

is an absolutely convergent series, since it converges when all terms are given the positive sign.

126. Tests of convergence. To test the convergence of a series, certain criteria are necessary. It is to be remembered that whether the terms of the series are constants or functions of a variable, we are concerned only with the limit of the sum of a finite number of constant terms as that number increases indefinitely. For, in case the terms are functions of a variable, we either substitute a constant value for the variable and then pass to the limit, or, what amounts to the same thing, we consider for what values of the variable the series has a limit. It is important, however, to distinguish between convergence at a point, and convergence throughout an interval.

The following tests, already considered in algebra, and consequently needing no proof here, will be found sufficient for series which arise in ordinary practice.*

THEOREM I. *Comparison test. Given a series of positive terms*

$$u_1 + u_2 + u_3 + \dots + u_n + \dots$$

If from some point on, the terms of this series are never greater than the corresponding terms of a known convergent series

$$v_1 + v_2 + v_3 + \dots + v_n + \dots$$

of positive terms, then the given series is convergent. If the terms of the given series are from some point on never less than the corresponding terms of a known divergent series of positive terms

$$t_1 + t_2 + t_3 + \dots + t_n + \dots,$$

then the given series is divergent.

Ex. 1. Test the convergency of the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!} + \dots \tag{1}$$

From Ex. 1 of the previous article, we have the convergent series ($|x| < 1$)

$$1 + x^2 + x^3 + x^4 + \dots + x^n + \dots \tag{2}$$

For $x = \frac{1}{2}$, this becomes

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots \tag{3}$$

Since for $n > 2$, $2^{n-1} < n!$, each term of (1) after the second is less than the corresponding term of (3); therefore the series (1) must converge.

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* See Osgood's *Infinite Series*, Arts. 5, 8, 9.

Ex. 2. Given the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2n-1} + \ldots \tag{4}$$

Compare this series with the series whose first n terms are

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n} = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right). \tag{5}$$

By Ex. 2 of the previous article, the series (5) is divergent. Moreover, since $2n - 1 < 2n$, each term of the given series (4) is greater than the corresponding term of the series defined by (5); consequently, the series (4) is divergent.

THEOREM II. *Ratio test. A given series*

$$u_1 + u_2 + \ldots + u_{n-1} + u_n + \ldots$$

is convergent or divergent, according as the limit $\underset{n=\infty}{L} \dfrac{u_n}{u_{n-1}}$ *is less or greater than* 1.

Ex. 3. Given the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \ldots \tag{6}$$

We have then

$$\underset{n=\infty}{L} \frac{u_n}{u_{n-1}} = \underset{n=\infty}{L} \frac{\dfrac{x^n}{n!}}{\dfrac{x^{n-1}}{(n-1)!}} = \underset{n=\infty}{L} \frac{x}{n} = 0,$$

for any finite value of x. Hence the series converges for all finite values of x.

It is to be noted that this second criterion applies equally well when some of the terms of the series are negative. It gives no test, however, when the limit of the ratio $\dfrac{u_n}{u_{n-1}}$ is 1; in such cases other methods must be employed.

Since each term of a series is finite, the sum of any finite number of terms is also finite, and consequently the convergency or divergency of a series is not affected by omitting a finite number of terms.

If the terms of the given series are alternately positive and negative, the following is a convenient test of convergence.

THEOREM III. *An infinite series in which the terms are alternately positive and negative is convergent if its terms decrease numerically and approach zero as a limit when n increases indefinitely.*[*]

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[*] For proof of this theorem, see Osgood's *Infinite Series*, Art. 11.

Ex. 4. Given the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

This series converges since it fulfills the conditions of the theorem. As the series formed by taking all the terms with positive signs does not converge, the given series converges conditionally.

EXERCISES.

Test the convergency or divergency of the following series. When the terms involve a variable, determine the interval of convergence.

1. $x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} - \dots$
2. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$
3. $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$
4. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
5. $\frac{1}{1+x^2} + \frac{1}{1+2x^2} + \frac{1}{1+3x^2}, +$
6. $\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots$

127. Uniform convergence. As we have already seen, a series of terms containing a variable is said to converge in a given interval if it converges for each constant value of the variable in that interval. Such a series represents a function in the interval of convergence, which may be plotted just as any other function.

Closely associated with the curve representing the function defined by the series is the succession of curves representing the sum of some finite number of terms, namely, the curves representing geometrically the relations

$$y = S_n(x), \quad n = 1, 2, 3, \dots$$

These curves are called the *approximation curves* of the given series. They may, and in general will, change as n changes. As n increases without limit, they may or may not approach the form of the curve representing the function defined by the given series. This will be seen more clearly in connection with the examples discussed later.

If the approximation curves of the series *do* approach as a limit the curve representing the function defined by the series, we say that the series *converges uniformly* within the given interval. The statement that a series converges uniformly contains much more than the simple statement that it converges in the ordinary sense, and ascribes to the series certain important properties that it would not otherwise possess.

Geometrically, uniform convergence means that having plotted the curve $y = f(x) = \underset{n=\infty}{L} S_n(x)$, and having laid off on each side of this curve an arbitrarily narrow strip, say of width σ (Fig. 118), it is always possible to find a value of n such that for *all* larger values the corresponding approximation curves shall lie within this strip. It is important for the student to notice the *order* of procedure. First of all, the curve representing the function defined by the series is plotted. Then the parallel strip is determined by the selection of σ, which is taken as small as we please. Finally, in order to have uniform convergence, it must be possible to find some value of n such that the curves $y = S_n(x)$ will always fall within this strip as n takes still larger values. As will be seen, the approximation curves will in this case approach as a limit the curve representing $f(x)$ as the definition requires. Uniform convergence differs then from ordinary convergence in that $S_n(x)$ approaches $f(x)$ simultaneously for all values of x rather than for each constant value of x in the given interval. As we shall see, this gives us a very important property of the series not possessed by the non-uniformly convergent series.

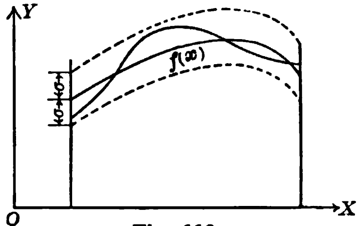

Fig. 118.

The distinction between ordinary convergence and uniform convergence is illustrated by the following examples.

Ex. 1. Given the series

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots,$$

where $0 \leqq x \leqq 1$.

This series converges for all values of x within this interval, and represents the function arc tan x. The curve

$$y = f(x) = \text{arc tan } x$$

and several of the approximation curves are given in Fig. 119. It will be seen by inspection that if we select a strip on either side of the curve $y = f(x)$, it is always possible to choose n so large that the corresponding approximation curve and all succeeding curves shall lie within this strip. Hence the series converges uniformly.

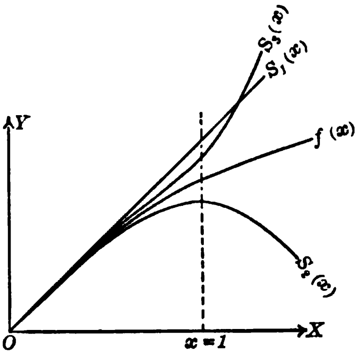

Fig. 119.

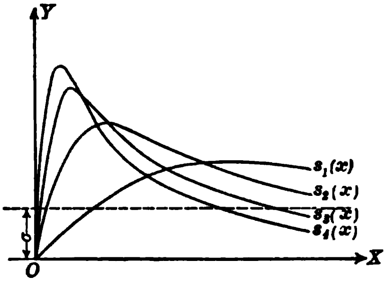

Fig. 120

Ex. 2. Given the series, the sum of whose first n terms is

$$S_n(x) = \frac{n^2x}{1 + n^3x^2}, \qquad 0 \leqq x \leqq 1.$$

This series converges and represents the function

$$y = f(x) = 0.$$

The series does not converge uniformly in the given interval, for the approximation curves do not fall within an arbitrarily small strip on the XY-plane. There is always a peak present in the neighborhood of the origin, and this peak extends beyond the arbitrarily small strip. See Fig. 120.

The following theorem gives a convenient test for uniform convergence, and will be found sufficient for the series to be considered in the present volume.

THEOREM. *Form a new series of positive terms by taking the numerically greatest value which each term of the given series may have in the interval* (α, β). *If this series converges, then the given series converges uniformly in the interval* (α, β).

Let the given series be

$$f(x) = u_1(x) + u_2(x) + \ldots + u_k(x) + \ldots + u_n(x) + \ldots \qquad (1)$$

Denote the numerically greatest value of the term $u_k(x)$ by M_k. Then by hypothesis the series

$$M_1 + M_2 + \dots + M_k + \dots + M_n + \dots \qquad (2)$$

converges in the interval (α, β). Denote the sum of the remaining terms of series (2) after the first n terms by P_n, and let $R_n(x)$ be the corresponding remainder of (1). Each term of series (1) is numerically less than, or at most equal to, the corresponding term of (2). Hence, for all values of x in (α, β), we have

$$| R_n(x) | \leqq P_n. \qquad (3)$$

However, since the series (2) converges, it follows that for n greater than some integer m, the remainder P_n is less than an arbitrarily chosen positive number σ. Hence we have

$$| R_n(x) | \leqq \sigma, \qquad n > m,$$

or, what is the same thing,

$$| f(x) - S_n(x) | \leqq \sigma, \quad n > m,$$

independently of what may be the value of x in the interval (α, β). This, however, is nothing else than the definition of uniform convergence in the given interval.

Ex. **3.** Given the series

$$\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots, \quad -\pi \leqq x \leqq \pi.$$

Forming the series of the numerically greatest values of the terms of this series, we have

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots,$$

which is term by term less than the known convergent series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Consequently, the given series converges uniformly in the interval $(-\pi, +\pi)$.

EXERCISES.

Test for uniform convergence the following series:

1. $1 + x + x^2 + x^3 + \dots \qquad |x| < \frac{1}{2}$.

2. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$

which defines the function $\log(1 + x)$ for the interval $-1 < x \leqq 1$.

3. $S_n(x) = \frac{2nx}{1+n^2x^2}$. (Test by plotting approximation curves.)

128. Power series. By a power series we understand a series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots, \tag{1}$$

where $a_0, a_1, \ldots a_n$, etc. are constants. Such series are of importance in mathematics because of the frequency with which they occur and because of the special properties which they possess. Two of the most important of these properties are expressed in the following theorems.

THEOREM I. *If a power series converges for $x = \alpha$, it is absolutely convergent for all values of x such that $|x| < |\alpha|$.*

Let the given series be written in the form

$$a_0 + a_1\alpha\left(\frac{x}{\alpha}\right) + a_2\alpha^2\left(\frac{x}{\alpha}\right)^2 + \ldots + a_n\alpha^n\left(\frac{x}{\alpha}\right)^n + \ldots; \tag{2}$$

then the series of coefficients

$$a_0, a_1\alpha, a_2\alpha^2, \ldots, a_n\alpha^n, \ldots$$

must decrease indefinitely in absolute value since the given series converges for $x = \alpha$. Let M be equal to or greater than the absolute value of any number in this sequence. Then the absolute values of the terms of (2) are less than the corresponding terms of the geometric series

$$M\left(1 + \frac{x}{\alpha} + \frac{x^2}{\alpha^2} + \frac{x^3}{\alpha^3} + \ldots + \frac{x^n}{\alpha^n} + \ldots\right),$$

which converges for $|x| < |\alpha|$. Hence the given series (1) converges absolutely for $|x| < |\alpha|$.

THEOREM II. *If a given power series converges for $x = \alpha$, and β is any number numerically less than α, then the given series converges uniformly in the interval $(-\beta, \beta)$.*

The numerically greatest value of each term is found by giving to x its greatest value in the interval $(-\beta, \beta)$, namely, $x = \beta$. The series thus formed is what we should obtain by putting $x = \beta$ in the given series and taking the absolute value of each term. It follows from Theorem I that this series of numerically greatest values converges since the given series converges by hypothesis for $x = \alpha$. Hence, from the theorem given in the last article for testing uniform convergence, we may assert that the given series must converge uniformly in the interval $-\beta \leqq x \leqq \beta$.

129. Term-by-term integration of series. The series of integrals obtained by integrating the separate terms of a given series does not necessarily represent the integral of the function represented by the given series. When it does, however, the term-by-term integration often gives the easiest method of obtaining such an integral. Since the integral of the sum of a finite number of terms is the sum of their integrals, we may represent the term-by-term integration of an infinite series as follows:

$$\int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \dots = \underset{n=\infty}{L} \int_a^b S_n(x)\,dx.$$

On the other hand, the integral of the function represented by such a series is given by

$$\int_a^b [u_1(x) + u_2(x) + \dots]\,dx = \int_a^b \Big[\underset{n=\infty}{L} S_n(x)\Big]\,dx.$$

It may easily be shown geometrically that these two results are the same whenever the series converges uniformly and a and b lie in the interval of uniform convergence. We have seen that if a series converges uniformly, the approximation curves $y = S_n(x)$ approach as a limit the curve representing the limiting function $f(x)$ as n increases indefinitely. But the integral $\int_a^b S_n(x)dx$ represents the area between this curve and the X-axis, lying between the ordinates $x = a$ and $x = b$. As n increases, this area may change, but as the curve ultimately approaches the curve representing $y = f(x)$, the area underneath the approximation curve must approach as a limit the area underneath the curve $y = f(x)$; that is, the two integrals must be equal, as has been stated. Thus we have

THEOREM. *Any series of continuous functions which converges uniformly in the interval* (α, β) *may be integrated term by term if the limits of integration are finite and lie in the interval* (α, β).

This condition may be applied at once to power series, since such series converges uniformly. If the limits of integration a, b lie in the interval of uniform convergence of the given power series

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots, \tag{1}$$

we then may write

$$\int_a^b f(x)\,dx = \int_a^b a_0 dx + \int_a^b a_1 x\,dx + \ldots + \int_a^b a_n x^n\,dx + \ldots \tag{2}$$

130. Term-by-term differentiation of a series. Uniform convergence of a series enables us also to formulate a criterion for the term-by-term differentiation of a series. This criterion may be stated as follows:

THEOREM. *A convergent series of functions may be differentiated term by term if the series resulting from the differentiation converges uniformly and its terms are continuous functions.*

Let $f(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$

be a convergent series in the interval (α, β), and let us put

$$\phi(x) = u_1'(x) + u_2'(x) + \ldots + u_n'(x) + \ldots \tag{1}$$

We are to show that $\phi(x) = f'(x)$. Since the terms of (1) are continuous and the series converges uniformly, we have for $\alpha < x < \beta$,

$$\begin{aligned}\int_a^x \phi(x)\,dx &= \int_a^x u_1'(x)\,dx + \int_a^x u_2'(x)\,dx + \ldots \\ &\qquad + \int_a^x u_n'(x)\,dx + \ldots \\ &= u_1(x) - u_1(\alpha) + \ldots + u_n(x) - u_n(\alpha) + \ldots \\ &= f(x) - f(\alpha). \end{aligned} \tag{2}$$

Since $f(\alpha)$ is a constant, and differentiation is the inverse of integration, we have as the result of differentiating (2),

$$\phi(x) = f'(x),$$

which establishes the theorem.

If we differentiate the terms of a power series, we have again a power series whose terms are continuous functions of x. If this resulting power series converges, it follows from the above theorem and Theorem II of Art. 128 that within the proper limits for the variable it represents the derivative of the function defined by the given series.

131. Expansion of functions in series. In the first chapter it was pointed out that certain classes of functions cannot be expressed algebraically in terms of the independent variable without resorting to infinite series or some other infinite process. Other functions may be expressed by means of infinite series, and often it is convenient to do so. We shall discuss methods of doing this by making use of the principles of the differential calculus.

In Art. 126, we saw that a convergent series may define a function in the interval of convergence; that is, a function $f(x)$ is defined by the relation

$$\underset{n=\infty}{L} S_n(x) = f(x)$$

for all values of x for which the series converges. This does not say that the series defines a particular function. In other words, no restriction is placed upon the character of the function $f(x)$ except that which the character of the particular series imposes. We are now confronted by a very different question. Given a particular function of x, say $f_1(x)$; what series will represent it? Moreover, will the interval in which this series represents the given function correspond to the interval in which the series converges? Certainly it cannot exceed this interval of convergence, since an infinite series cannot be said to represent a function in the ordinary sense unless it converges.

Let us denote the difference between the given function $f_1(x)$ and the sum of the first n terms of the series by $R_n(x)$. We may then write

$$f_1(x) = S_n(x) + R_n(x).$$

All that is necessary in order that the given series shall converge and therefore represent some function, is that the limit $\underset{n=\infty}{L} S_n(x)$ shall exist for the values of x in question. In order that the series shall represent the given function $f_1(x)$, we must have also

$$\underset{n=\infty}{L} R_n(x) = 0.$$

It is entirely conceivable that for some values of x this last limit may be different from zero, and still the two limits $\underset{n=\infty}{L} S_n(x)$ and $\underset{n=\infty}{L} R_n(x)$ each exist. In such a case the series clearly does not represent the given function.

The interval for which the limit $\underset{n=\infty}{L} R_n(x) = 0$, that is the interval in which the series represents the given function, is called the *interval of equivalence*. This may be less than or equal to the interval of convergence of the given series. In most cases with which we shall have to deal, the two are identical. The distinction, however, is of importance in discussing the expansion of a function.

The distinction between the interval of equivalence and the interval of convergence may be illustrated by making use of the geometrical representation of the convergence of a series. Suppose that $S_n(x)$ converges toward the values represented by the continuous curve $y = f(x)$ for $0 \leqq x \leqq \beta$, and the given function $f_1(x)$ is represented by the dotted line Fig. 121. Then for a value like $x = x_0$, the series converges toward a value which is at the same time a value of the given function. On the other hand, for a value like $x = x_1$, the series converges toward a value $f(x_1)$ which is different from the value of the given function for $x = x_1$, namely, $f_1(x_1)$. Hence, while the interval of convergence is $(0, \beta)$, the interval of equivalence is only (α, β).

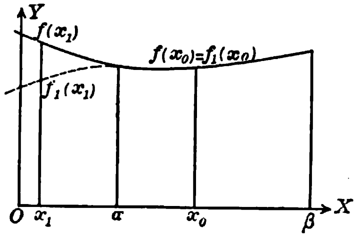

Fig. 121.

132. Maclaurin's expansion of a function in a power series. Suppose we have given a function $f(x)$ which, together with its derivatives, is continuous in the vicinity of the value $x = 0$. If such a function can be represented by a power series, that series must be of the form

$$f(x) = A_0 + A_1x + A_2x^2 + \dots + A_nx^n + \dots, \tag{1}$$

where the coefficients $A_0, A_1, A_2, \dots, A_n, \dots$ are to be determined. Since this is a power series, we may find the successive derivatives of the function $f(x)$ by term-by-term differentiation of the series. The following identities are thus derived:

$$\begin{aligned}
f'(x) &= A_1 + 2\,A_2x + 3\,A_3x^2 + 4\,A_4x^3 + \dots \\
f''(x) &= 2\,A_2 + 3 \cdot 2\,A_3x + 4 \cdot 3\,A_4x^2 + \dots \\
f'''(x) &= 3 \cdot 2\,A_3 + 4 \cdot 3 \cdot 2\,A_4x + \dots \\
f^{\text{iv}}(x) &= 4 \cdot 3 \cdot 2\,A_4 + \dots \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

By hypothesis, each of these expansions may be held as identical with the functions they represent for any value of x within the interval of equivalence, which in this case includes the value $x = 0$. Substituting $x = 0$ in each of the above identities, we have

$$f(0) = A_0,\quad f'(0) = A_1,\quad f''(0) = 2!\,A_2,\quad f'''(0) = 3!\,A_3, \ldots$$

Hence the successive coefficients are

$$A_0 = f(0),\quad A_1 = f'(0),\quad A_2 = \frac{f''(0)}{2!},\quad A_3 = \frac{f'''(0)}{3!}, \ldots \quad (2)$$

We have thus the values of the unknown coefficients in the assumed expansion in terms of the successive derivatives of the given function. Substituting these values in that expansion, we have

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^n(0)}{n!}x^n + \ldots, \quad (3)$$

where $f^n(0)$ denotes the result obtained by differentiating the function $f(x)$ in succession n times and substituting $x = 0$ in the result. The above series is called *Maclaurin's series.*

We have still to determine the interval of equivalence of the given function and the series obtained. This is accomplished by examining the limit of the remainder $R_n(x)$ as n increases indefinitely. We shall discuss the form and character of $R_n(x)$ in a subsequent article. The following examples will illustrate the method of computing the successive coefficients in the required expansion. When the successive derivatives all become zero from some point on, the expansion has a finite number of terms.

Ex. 1. Expand $f(x) = (1 + x)^m$.

We have

$$\begin{aligned} f(x) &= (1+x)^m, \\ f'(x) &= m(1+x)^{m-1}, \\ f''(x) &= m(m-1)(1+x)^{m-2}, \\ &\ldots\ldots\ldots\ldots \\ f^n(x) &= m(m-1)\ldots(m-n+1)(1+x)^{m-n}, \end{aligned}$$

whence

$$f(0) = 1,\quad f'(0) = m,\quad f''(0) = m(m-1), \text{ etc.}$$

Substituting these values in (1), we have, when m is negative or fractional, the infinite series

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots$$
$$+ \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \dots,$$

which converges for $|x| < 1$.

If m is a positive integer, the series terminates with first $m + 1$ terms, since all of the higher derivatives vanish.

Ex. **2.** Expand $\sin x$ in a power series.

$$\begin{array}{ll} f(x) = \sin x, & f(0) = \sin 0 = 0, \\ f'(x) = \cos x, & f'(0) = 1, \\ f''(x) = -\sin x, & f''(0) = 0, \\ f'''(x) = -\cos x, & f'''(0) = -1, \\ f^{\text{IV}}(x) = \sin x, & f^{\text{IV}}(0) = 0, \\ \text{etc.} & \text{etc.} \end{array}$$

Hence

$$\sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \dots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

The interval of convergence for this series is $(-\infty, +\infty)$.

EXERCISES.

Expand in power series the following functions, assuming that such expansions are possible, and determine the interval of convergence in each case.

1. e^x.
2. $\cos x$.
3. a^x.
4. $e^{\sin x}$.
5. $\log(1 + e^x)$.
6. $\log(1 + x)$.
7. $\text{arc} \sin x$.
8. $\text{arc} \tan x$.
9. $(\cos\theta)^n$.
10. $e^\theta \sec\theta$.
11. $\sec\theta$ (to four terms).
12. $e^{\text{arc} \sin \theta}$.
13. $\log(x + \sqrt{1 + x^2})$.
14. $\log\cos\theta$.
15. $\cosh u = \frac{1}{2}(e^u + e^{-u})$.
16. $\log(1 - x + x^2)$.

133. Taylor's expansion. Thus far we have studied the expansion of a given function $f(x)$ in the vicinity of the value $x = 0$. The method may be easily extended to the expansion in the neighborhood of any point $x = a$, provided the given function and its successive derivatives are continuous. All that is necessary is to assume the expansion of the form

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \dots + A_n(x-a)^n + \dots \quad (1)$$

and proceed precisely as in Art. 132. The resulting form of the expansion is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$
$$+ \frac{f^n(a)}{n!}(x-a)^n + \dots \qquad (2)$$

which holds for all values of the variable within the interval of equivalence. This expansion is known as *Taylor's series.*

If in (2) we replace x by $(x + a)$, we have

$$f(x + a) = f(a) + f'(a)\,x + \frac{f''(a)}{2!}\,x^2 + \dots + \frac{f^n(a)}{n!}\,x^n + \dots, \qquad (3)$$

which is a form in which Taylor's series is frequently written. If x and a are interchanged, the expansion takes the form

$$f(a + x) = f(x) + f'(x)\,a + \frac{f''(x)}{2!}\,a^2 + \dots + \frac{f^n(x)}{n!}\,a^n + \dots \qquad (4)$$

Forms (3) and (4) are useful when it is desired to expand a function of the sum of two numbers in powers of one of them.

Ex. 1. Expand e^x in powers of $x - 1$.

$$\begin{aligned} f(x) &= e^x, & f(1) &= e,\\ f'(x) &= e^x, & f'(1) &= e,\\ f''(x) &= e^x, & f''(1) &= e,\\ &\text{etc.} & &\text{etc.} \end{aligned}$$

Hence

$$e^x = e\left[1 + (x - 1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots\right].$$

Ex. 2. Express $3x^3 - 5x^2 + 8x - 5$ in powers of $x - 2$.

In this case

$$\begin{aligned} f(x) &= 3x^3 - 5x^2 + 8x - 5, & f(2) &= 15,\\ f'(x) &= 9x^2 - 10x + 8, & f'(2) &= 24,\\ f''(x) &= 18x - 10, & f''(2) &= 26,\\ f'''(x) &= 18, & f'''(2) &= 18,\\ f^{\text{IV}}(x) &= 0, & f^{\text{IV}}(2) &= 0. \end{aligned}$$

Hence we have

$$f(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3,$$

or

$$3x^3 - 5x^2 + 8x - 5 = 15 + 24(x - 2) + 13(x - 2)^2 + 3(x - 2)^3.$$

Ex. 3. Develop $\log(x + h)$ in powers of h.

$$f(x+h) = \log(x+h), \qquad f(x) = \log x.$$

Hence

$$f'(x) = \frac{1}{x}, \qquad f''(x) = -\frac{1}{x^2},$$

$$f'''(x) = \frac{1 \cdot 2}{x^3}, \qquad f^{\text{IV}}(x) = -\frac{1 \cdot 2 \cdot 3}{x^4},$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$f^n(x) = -(-1)^n \frac{1 \cdot 2 \ldots (n-1)}{x^n}.$$

Substituting in (3), we obtain

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \ldots$$

For $x = 1$, we have

$$\log(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \ldots$$

Ex. 4. Expand $\sin(x + y)$ and derive the formula

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

We have

$$\begin{aligned} f(x) &= \sin x, \\ f'(x) &= \cos x, \\ f''(x) &= -\sin x, \\ f'''(x) &= -\cos x, \text{ etc.} \end{aligned}$$

Hence

$$\begin{aligned} \sin(x+y) &= \sin x + y \cos x - \frac{y^2}{2!}\sin x - \frac{y^3}{3!}\cos x + \frac{y^4}{4!}\sin x + \frac{y^5}{5!}\cos x - \ldots \\ &= \sin x\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots\right) \\ &\qquad + \cos x\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \ldots\right) \\ &= \sin x \cos y + \cos x \sin y. \end{aligned}$$

Ex. 5. If $f(x) = 5x^3 - 4x^2 + 18x - 7$, find $f(x - 2)$ by Taylor's expansion.

We have here

$$\begin{aligned} f'(x) &= 15x^2 - 8x + 18, \\ f''(x) &= 30x - 8, \\ f'''(x) &= 30, \\ f^{\text{IV}}(x) &= 0. \end{aligned}$$

Hence from (1)

$$\begin{aligned} f(x-2) &= f(x) + (-2)f'(x) + \frac{(-2)^2}{2!}f''(x) + \frac{(-2)^3}{3!}f'''(x) \\ &= 5x^3 - 4x^2 + 18x - 8 \\ &\qquad - 30x^2 + 16x - 36 \\ &\qquad + 60x - 16 \\ &\qquad - 40 \\ &= 5x^3 - 34x^2 + 94x - 99. \end{aligned}$$

EXERCISES.

Assuming that Taylor's expansion is possible, develop the following functions.

1. e^{x+h}.
2. $(x+y)^m$.
3. $(x+y)^6$.
4. arc sin $(x+h)$ to four terms.
5. log sin $(x+h)$.
6. Find $f(x+3)$, when $f(x) = x^3 - 4x + 7$.
7. Find $f(x-1)$, when $f(x) = x^2 + 7x - 5$.
8. Show that $\log(x+\sqrt{1+x^2}) = x - \frac{x^3}{2\cdot 3} + \frac{3x^5}{2\cdot 4\cdot 5} - \dots$
9. Expand sinh $(x+y)$.
10. Expand cosh $(x+y)$.
11. Expand sin x in powers of $x-a$.
12. Express $5x^3 - 6x^2 + x - 10$ in powers of $x-1$; also in powers of $x-3$. Verify the results.
13. Expand log x in powers of $x-1$. Find the interval of convergence.
14. Expand $\frac{1}{x}$ in powers of $x-a$ and determine the interval of convergence.

134. Form of remainder for Taylor's series. The forms of expansion discussed in the preceding articles hold only for the interval of equivalence of the function and derived series. This cannot be greater than the interval of convergence of the series. The interval of equivalence is determined by finding for what values of the variable the remainder has the limit zero, where by the remainder we mean the difference between the given function and the sum of the first n terms of the series. The remainder, denoted as heretofore by R_n, may be written in the form

$$R_n(x, a) = \frac{(x-a)^n}{n!}\phi(x, a). \tag{1}$$

We must now so determine $\phi(x, a)$ that the following relation holds identically

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$
$$+ \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \frac{\phi(x, a)}{n!}(x-a)^n. \tag{2}$$

Transposing, we have

$$f(x) - f(a) - f'(a)(x-a) - \dots - \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1}$$
$$- \frac{\phi(x, a)}{n!}(x-a)^n = 0. \tag{3}$$

To determine the character of $\phi(x, a)$, let us set up arbitrarily the function

$$F(z) = f(x) - f(z) - f'(z)(x - z) - \frac{f''(z)}{2!}(x - z)^2 - \dots$$
$$- \frac{f^{n-1}(z)}{(n-1)!}(x - z)^{n-1} - \frac{\phi(x, a)}{n!}(x - z)^n. \tag{4}$$

The function $F(z)$ fulfills the conditions of Rolle's theorem; it possesses a derivative, vanishes for $z = a$ by virtue of (3) and for $z = x$ by inspection. Hence the derivative must vanish for some value of z lying between a and x, say z_1. Differentiating with respect to z, we have

$$F'(z) = -f'(z) + f'(z) - f''(z)(x - z) + f''(z)(x - z) - \dots$$
$$- \frac{f^n(z)}{(n-1)!}(x - z)^{n-1} + \frac{\phi(x, a)}{(n-1)!}(x - z)^{n-1}. \tag{5}$$

The terms of the right-hand member of this equation combine in pairs, and the result reduces to

$$F'(z) = \frac{(x - z)^{n-1}}{(n - 1)!}[\phi(x, a) - f^n(z)]. \tag{6}$$

Since the derivative $F'(z)$ must vanish for $z = z_1$, we have

$$\phi(x, a) = f^n(z_1).$$

By hypothesis z_1 lies between x and a; hence we may write

$$z_1 = a + \theta(x - a), \tag{7}$$

where $0 < \theta < 1$. Therefore $\phi(x, a)$ may be written in the form

$$\phi(x, a) = f^n[a + \theta(x - a)], \tag{8}$$

and the remainder R_n becomes

$$R_n(x, a) = \frac{f^n[a + \theta(x - a)]}{n!}(x - a)^n. \tag{9}$$

We have, therefore, the following formula for Taylor's expansion when written in finite form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$
$$+ \frac{f^{n-1}(a)}{(n - 1)!}(x - a)^{n-1} + \frac{f^n[a+\theta(x-a)]}{n!}(x-a)^n. \tag{10}$$

This series can be extended to any number of terms. The last term always gives the remainder; that is, the error in omitting the remaining terms of the series. When the limit of this remainder is zero as n becomes infinite, the function is represented by the corresponding infinite series.

Instead of writing the remainder in the form given by (1), it is frequently convenient to put it in the form

$$R_n(x, a) = (x - a)\,\chi(x, a). \tag{11}$$

If we accept this form and proceed as before, we have

$$F'(z) = -\frac{f^n(z)}{(n-1)!}(x - z)^{n-1} + \chi(x, a). \tag{12}$$

Since $F'(z)$ vanishes for $z = z_1$, we have

$$\chi(x, a) = \frac{f^n(z_1)}{(n-1)!}(x - z_1)^{n-1}, \tag{13}$$

where $z_1 = a + \theta(x - a)$, or $x - z_1 = (x - a)(1 - \theta)$. Consequently, we may write

$$\chi(x, a) = \frac{f^n[a + \theta(x - a)]}{(n-1)!}(1 - \theta)^{n-1}(x - a)^{n-1}, \tag{14}$$

and it follows immediately that

$$R_n(x, a) = (1 - \theta)^{n-1}\frac{f^n[a + \theta(x - a)]}{(n-1)!}(x - a)^n. \tag{15}$$

When Taylor's expansion is written in the form given by (3) or (4) of Art. 133, the remainder becomes

$$R_n(x, a) = \frac{f^n(a + \theta x)}{n!}x^n, \quad \text{or} \quad \frac{f^n(x + a\theta)}{n!}a^n, \tag{16}$$

a result which the student can easily obtain from the general form of the remainder.

It is often convenient to make use of the second form of the remainder. For form (3) this is

$$R_n(x, a) = (1 - \theta)^{n-1}\frac{f^n(a + \theta x)}{(n-1)!}x^n, \tag{17}$$

and for form (4) it becomes

$$R_n(x, a) = (1 - \theta)^{n-1}\frac{f^n(x + a\theta)}{(n-1)!}a^n. \tag{18}$$

135. Form of remainder for Maclaurin's series. The special form of the remainder for Maclaurin's series may be easily obtained by putting $a = 0$ in the result found in the previous article. The result is

$$R_n(x, 0) = \frac{f^n(\theta x)}{n!} x^n, \tag{1}$$

and consequently Maclaurin's expansion written in finite form is

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} + \\ + \frac{f^n(\theta x)}{n!}x^n. \tag{2}$$

Whenever in any particular expansion the limit

$$\underset{n=\infty}{L} \frac{f^n(\theta x)}{n!} x^n$$

is equal to zero for any interval, then that interval is the interval of equivalence, within which the function is represented by the series in question.

The following example illustrates the method of determining the interval of equivalence.

Ex. Investigate the remainder in the expansion for $\sin x$.

From Ex. 2, Art. 132, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^n}{n!}f^n(\theta x).$$

But $$f^n(x) = \sin\left(n\frac{\pi}{2} + x\right),$$

whence $$f^n(\theta x) = \sin\left(n\frac{\pi}{2} + \theta x\right),$$

and therefore $$R_n = \frac{x^n}{n!}\sin\left(n\frac{\pi}{2} + \theta x\right).$$

Since $\sin\left(n\frac{\pi}{2} + \theta x\right) \leqq 1$ and $\underset{n=\infty}{L}\frac{x^n}{n!} = 0$, it follows that $\underset{n=\infty}{L} R_n = 0$ for all values of x within the interval $(-\infty, +\infty)$. Hence the interval of equivalence is in this case the same as the interval of convergence.

EXERCISES.

For each of the following functions find the form of the remainder and determine the interval of equivalence.

1. e^x. 2. $\cos x$. 3. $\log(1 + x)$. 4. a^x.

136. Extension of Taylor's expansion to functions of two variables. In a manner similar to that employed in the previous articles, we may develop a form of expansion for a function of two variables.

Let $f(x, y)$ and its successive partial derivatives be continuous functions with respect to both variables within the region under consideration. It is required then to express $f(x + h, y + k)$ in ascending powers of h and k. Replace h and k by ht and kt, respectively, and suppose the variables to have initial values x_0, y_0; then the function to be developed is $f(x + ht, y + kt)$. We may regard $f(x_0 + ht,\ y_0 + kt)$ as a function of t and denote it by $F(t)$. Making use of Maclaurin's expansion, we have

$$F(t) = F(0) + F'(0)t + \frac{F''(0)}{2!}t^2 + \ldots + \frac{F^{n-1}(0)}{(n-1)!}t^{n-1} + \frac{F^n(\theta t)}{n!}t^n. \tag{1}$$

If we now put $t = 1$, (1) becomes

$$F(1) = f(x_0+h,\ y_0+k) = F(0) + F'(0) + \frac{F''(0)}{2!} + \ldots$$
$$+ \frac{F^{n-1}(0)}{(n-1)!} + \frac{F^n(\theta)}{n!}, \tag{2}$$

in which the successive derivatives of F for the value $t = 0$ must be replaced by corresponding expressions in terms of $f(x, y)$. For this purpose we have

$$F(t) = f(x_0 + ht, y_0 + kt),$$

whence

$$F'(t) = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{3}$$

By means of the equations $x = x_0 + ht$, $y = y_0 + kt$, we obtain

$$\frac{dx}{dt} = h, \qquad \frac{dy}{dt} = k,$$

whence (3) becomes

$$F'(t) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}.$$

A second differentiation gives

$$\begin{aligned} F''(t) &= h\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) + k\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) \\ &= h\left[\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y\,\partial x}\frac{dy}{dt}\right] + k\left[\frac{\partial^2 f}{\partial y\,\partial x}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right] \\ &= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial y\,\partial x} + k^2\frac{\partial^2 f}{\partial y^2}. \end{aligned} \tag{4}$$

In a similar manner we obtain

$$F'''(t) = h^3\frac{\partial^3 f}{\partial x^3} + 3\,h^2k\,\frac{\partial^3 f}{\partial x^2\,\partial y} + 3\,hk^2\,\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\,. \tag{5}$$

From these values, we have at once the coefficients of (1) when we replace t by 0. Thus

$$\begin{aligned}
F(0) &= f(x_0, y_0),\\
F'(0) &= h\,\frac{\partial f}{\partial x} + k\,\frac{\partial f}{\partial y},\\
F''(0) &= h^2\,\frac{\partial^2 f}{\partial x^2} + 2\,hk\,\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\,,\\
F'''(0) &= h^3\,\frac{\partial^3 f}{\partial x^3} + 3\,h^2k\,\frac{\partial^3 f}{\partial x^2\,\partial y} + 3hk^2\,\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\,\frac{\partial^3 f}{\partial y^3}\,.\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

The general form of the expansion is therefore

$$\begin{aligned}
f(x + h, y + k) &= f(x, y) + \left(h\,\frac{\partial f}{\partial x} + k\,\frac{\partial f}{\partial y}\right)\\
&+ \frac{1}{2!}\left(h^2\,\frac{\partial^2 f}{\partial x^2} + 2\,hk\,\frac{\partial^2 f}{\partial x\,\partial y} + k^2\,\frac{\partial^2 f}{\partial y^2}\right)\\
&+ \frac{1}{3!}\left(h^3\,\frac{\partial^3 f}{\partial x^3} + 3\,h^2k\,\frac{\partial^3 f}{\partial x^2\partial y} + 3\,hk^2\,\frac{\partial^3 f}{\partial x\partial y^2} + k^3\,\frac{\partial^3 f}{\partial y^3}\right) + \cdots
\end{aligned} \tag{6}$$

The succeeding terms may easily be written by observing that the exponents of h and k and the coefficients follow the law of the binomial expansion.

The form of the remainder after n terms of the expansion is readily shown to be

$$\frac{F^n(\theta)}{n!} = \frac{1}{n!}\sum_{r=0}^{n}\frac{n!}{r!\,(n-r)!}\,h^{n-r}k^r\,\frac{\partial^n f(x + \theta h, y + \theta k)}{\partial x^{n-r}\,\partial y^r}\,. \tag{7}$$

As in the case of functions of a single variable, the infinite expansion holds only when the limit of the remainder vanishes as n increases indefinitely.

EXERCISES.

1. If $f(x, y) = x^3\,(a + y)^2$, find $(x + h)^3\,(a + y + k)^2$.

2. Given $f(x, y) = ax^2 + bxy + cy^2 + dx + gy + m$. Expand $f(x+h, y+k)$ in powers of h and k.

3. Derive an approximation for $(x + h)\,(y + k)^n$ when h and k are small relatively to x and y. Verify by a numerical example.

4. Given $f(x, y) = e^x \sin y$, expand $f(x + h, y + k)$ in powers of h and k.

137. Expansions obtained by the integration and differentiation of series. It is sometimes possible to expand a function into an infinite series by means of term-by-term integration or differentiation of a known series. The restrictions of Arts. 129 and 130 must of course be observed.

Ex. 1. For $-1 < x < 1$,

we have
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

Hence
$$\int_0^x \frac{dx}{1+x} = \int_0^x dx - \int_0^x x\,dx + \int_0^x x^2\,dx - \int_0^x x^3\,dx + \dots,$$

or
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Ex. 2. For $-1 < x < 1$,
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

Differentiating both members, we obtain
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

Likewise
$$\frac{1}{(1-x)^3} = \frac{1}{2}(1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \dots),$$

and in general
$$\frac{1}{(1-x)^m} = (1-x)^{-m} = 1 + mx + \frac{m(m+1)}{2!}x^2 + \frac{m(m+1)(m+2)}{3!}x^3 + \dots$$

EXERCISES.

1. From the known series
$$1 - x^2 + x^4 - x^6 + \dots,$$
which defines the function $\frac{1}{1+x^2}$ for $-1 < x < 1$, derive the series for arc tan x.

2. From the series $1 + x^2 + x^4 + x^6 + \dots$, derive the series for inv. tanh x. Determine the interval of convergence.

3. For $-1 < x < 1$,
$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \dots$$
Derive from this relation a series for arc sin x.

4. Derive the series for cos x by differentiating that for sin x.

138. Use of series in computation. Infinite series may be used advantageously in the computation of certain constants, logarithms, trigonometric functions, and roots of numbers; also in the derivation of certain useful approximations.

I. *Computation of e.*

We have
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad (1)$$

Therefore for $x = 1$,
$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots, \qquad (2)$$
whence taking a sufficient number of terms, $e = 2.7182818\ldots$

II. *Computation of* π.

From the expansion
$$\text{arc sin } x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots, \qquad (3)$$
which holds for $-1 < x < 1$, we get for $x = \frac{1}{2}$
$$\text{arc sin}\frac{1}{2} = \frac{\pi}{6} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} \cdot \left(\frac{1}{2}\right)^5 + \ldots, \qquad (4)$$
whence
$$\pi = 3.14159\ldots$$

III. *Extraction of roots.*

We have
$$(a^n \pm b)^{\frac{1}{n}} = a\left(1 \pm \frac{b}{a^n}\right)^{\frac{1}{n}} = a\,(1 \pm x)^{\frac{1}{n}}, \qquad (5)$$
where $x = \frac{b}{a^n}$. Developing $(1 \pm x)^{\frac{1}{n}}$, we obtain
$$(1 \pm x)^{\frac{1}{n}} = 1 \pm \frac{1}{n}\,x - \frac{n-1}{n^2}\frac{x^2}{2!} \pm \frac{(n-1)(2n-1)}{n^3}\frac{x^3}{3!} - \ldots \qquad (6)$$

Ex. $\sqrt[5]{1000} = \sqrt[5]{1024 - 24} = 4\left(1 - \frac{3}{128}\right)^{\frac{1}{5}}$.

Substitute $\frac{3}{128}$ for x in the series
$$(1 - x)^{\frac{1}{5}} = 1 - \frac{x}{5} - \frac{4}{5}\frac{x^2}{10} - \frac{4}{5}\frac{9}{10}\frac{x^3}{15} - \ldots$$
The result of this substitution to six figures is 0.995268; hence
$$\sqrt[5]{1000} = 4 \times 0.995268 = 3.981072.$$

IV. *Computation of logarithms.*

The series for $\log(1 + x)$, that is,
$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots,$$
converges slowly, and is therefore not well adapted for computation.

A more useful series is derived as follows:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Substituting $-x$ for x, we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

By subtraction we get

$$\log(1+x) - \log(1-x) = \log\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right).$$

Let us take x positive and assume $x = \frac{1}{2y+1}$; then

$$\frac{1+x}{1-x} = \frac{y+1}{y},$$

and we have

$$\log(y+1) = \log y + 2\left[\frac{1}{2y+1} + \frac{1}{3}\left(\frac{1}{2y+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2y+1}\right)^5 + \dots\right] \tag{7}$$

From this series $\log(y+1)$ can be calculated when $\log y$ is known. Thus, since $\log 1 = 0$, we have

$$\begin{aligned}
\log 2 &= 2\left(\frac{1}{3} + \frac{1}{3}\frac{1}{3^3} + \frac{1}{5}\frac{1}{3^5} + \frac{1}{7}\frac{1}{3^7} + \dots\right)\\
&= 0.693147\dots\\
\log 3 &= \log 2 + 2\left(\frac{1}{5} + \frac{1}{3}\frac{1}{5^3} + \frac{1}{5}\frac{1}{5^5} + \dots\right)\\
&= 1.098612\dots\\
\log 4 &= 2\log 2 = 1.386294\dots\\
\log 5 &= \log 4 + 2\left(\frac{1}{9} + \frac{1}{3}\frac{1}{9^3} + \frac{1}{5}\frac{1}{9^5} + \dots\right)\\
&= 1.609438\dots\\
\log 6 &= \log 2 + \log 3 = 1.791759\dots
\end{aligned}$$

Evidently it is only necessary to make the computation in the case of prime numbers. Logarithms to the base 10 are obtained from the natural logarithms by means of the following relation:

$$\log_{10} a = \log_e a \times \frac{1}{\log_e 10} = 0.4342945\dots \cdot \log_e a.$$

V. *Computation of trigonometric functions.*

For all values of x, we have the series

$$\left.\begin{aligned} \sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \end{aligned}\right\} \tag{8}$$

These series converge rapidly, and may be used to compute the natural sine and cosine of any angle. Of course x must be expressed in radians.

Ex. Find the sine and cosine of 19° 30′ correct to five figures.

We have $x = \frac{19.5}{180}\pi = .34034$. Substituting this value for x in the two series, we get

$$\sin x = 0.33381,$$
$$\cos x = 0.94264.$$

The same method may be used to compute hyperbolic functions.

VI. *Approximation formulas.*

It is frequently convenient in computation to replace a function by another of approximately the same numerical value, but having a more simple form, or a form better adapted to calculation. This substitution may be effected in many cases by expanding the given function and taking a certain number of the first terms of the series. The following are among the most useful approximations:

1. Let m denote a small fraction. Then approximately

$$(1 \pm m)^n = 1 \pm nm. \tag{a}$$
$$\sqrt{1 \pm m} = 1 \pm \tfrac{1}{2} m. \tag{b}$$
$$e^m = 1 + m. \tag{c}$$
$$\log (1 + m) = m - \tfrac{1}{2} m^2. \tag{d}$$
$$\sin m = m\,(1 - \tfrac{1}{6} m^2). \tag{e}$$
$$\cos m = 1 - \tfrac{1}{2} m^2. \tag{f}$$

2. Let m be small in comparison with a. Then approximately

$$\sqrt{a^2 + m} = a\left(1 + \frac{m}{2\,a^2}\right). \tag{g}$$
$$\log (a + m) = \log a + \frac{m}{a} - \frac{m^2}{2\,a^2}. \tag{h}$$
$$\frac{1}{a \pm m} = \frac{1}{a} \mp \frac{m}{a^2} + \frac{m^2}{a^3}. \tag{i}$$
$$\sin (a \pm m) = \sin a \pm m \cos a. \tag{j}$$

The degree of error due to the neglected terms may be estimated by taking the maximum value of the remainder R. For example, the error in (e) due to taking the first two terms of the series is

$$R = \frac{m^5}{5!} f^{(5)}(\theta x) \leqq \frac{m^5}{120}.$$

If we wish to restrict the error to some definite limit r, we have only to write

$$| R | < r$$

and solve for m. Thus in the case just quoted, if we restrict the error to one unit in the fourth decimal place, we have

$$\left| \frac{m^5}{120} \right| < .0001,$$

whence $| m | < \sqrt[5]{.012}$, or $| m | < .413$. Hence for angles lying between $- 23° 40'$ and $+ 23° 40'$, two terms of the series give the value of $\sin m$ correct to three figures.

VII. *Approximations for circular arcs and chords.*

By the use of the sine and cosine series, convenient approximate relations between circular arcs and their chords may be obtained.

Let a circular arc of length s and radius r subtend a central angle α, and let C and c denote the chord of the arc and that of the half-arc, respectively, Fig. 122. We have then

$$s = r\alpha, \quad C = 2r \sin \frac{\alpha}{2}.$$

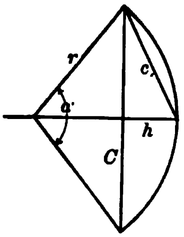

Fig. 122.

Using the series for $\sin \frac{\alpha}{2}$, we have

$$C = 2r\left(\frac{\alpha}{2} - \frac{\alpha^3}{48} + \frac{\alpha^5}{3840} - \ . \ . \ . \right). \tag{9}$$

If α is small, the terms of the series after the second may be neglected, whence the excess of the arc over the chord is approximately

$$s - C = \frac{r\alpha^3}{24}. \tag{10}$$

If α is measured in degrees, (10) becomes

$$s - C = \frac{r\alpha^3}{4514180}. \tag{11}$$

If the chord is given, and the arc is required, we may arrive at a less accurate formula as follows: For a small angle we may substitute C for s, whence approximately $\alpha = \frac{C}{r}$, and $r\alpha^3 = \frac{C^3}{r^2}$; then

$$s - C = \frac{C^3}{24\,r^2},$$

or
$$s = C\left[1 + \frac{1}{24}\left(\frac{C}{r}\right)^2\right]. \tag{12}$$

Huygens' approximation to the length of a circular arc is derived as follows: From Fig. 122, we have

$$\frac{C}{2r} = \sin\frac{\alpha}{2} = \frac{\alpha}{2} - \frac{\alpha^3}{2^3\cdot 3!} + \frac{\alpha^5}{2^5\cdot 5!} - \cdots,$$

$$\frac{c}{2r} = \sin\frac{\alpha}{4} = \frac{\alpha}{4} - \frac{\alpha^3}{4^3\cdot 3!} + \frac{\alpha^5}{4^5\cdot 5!} - \cdots$$

To get rid of the term in α^3, multiply the second equation by 8 and subtract the first. The result is

$$\frac{8c - C}{2r} = \frac{3}{2}\alpha - \frac{3\,\alpha^5}{2^7\cdot 5!} - \cdots,$$

whence
$$\frac{8c - C}{3} = r\alpha - \frac{r\alpha^5}{7680} - \cdots$$

Neglecting the term $\frac{r\alpha^5}{7680}$ and putting $r\alpha = s$, we have finally

$$s = \frac{8c - C}{3}. \tag{13}$$

The error of the approximation is less than the omitted term $\frac{r\alpha^5}{7680}$; hence the relative error is less than $\frac{\alpha^4}{7680.}$.

EXERCISES.

1. Calculate sin 15° and cos 15° to five decimal places.

2. From the series for tan x calculate tan 12°.

3. Find $\sqrt[3]{1334}$.

4. From the logarithms given in this article calculate log 31, also log 73.

5. Using the approximate formula (1) calculate the reciprocal of 102; of 97.

6. If approximation (g) is used in finding the value of the expression $\sqrt{L^2 - r^2\sin^2\theta}$, find the maximum error with $\frac{L}{r} = 6$.

7. Find the greatest value of m that will permit the approximation

$$(1 + m)^4 = 1 + 4m$$

with a maximum error of 1 in 1000.

8. Within what limits will three terms of the series for $\cos x$ give an error not exceeding 2 units in the 6th decimal place?

9. Investigate the limits of accuracy of the formula

$$\sin (a + m) = \sin a + m \cos a.$$

10. Find inv. tanh (.5) correct to five figures.

11. Find the length of the chord of an arc of radius 200 feet subtending an angle of 3°: (a) by trigonometric methods; (b) by the approximation formula. Compare the results and find the relative error of the approximation.

12. Find by Huygens' approximation the length of an arc of 30° in a circle of 10,000 feet radius. Compare with the true result, and find the relative error.

13. Find approximately the arc cut off by a chord of 2 inches long in a circle of 10-inch radius. Find the relative error.

14. Referring to Fig. 122, derive an approximate formula for the excess of the arc over the chord in terms of the chord and the dimension h.

139. Maxima and minima of functions of a single variable.

Taylor's expansion of a function gives a convenient method of developing the condition for maxima and minima. This method is particularly valuable where several of the derived functions vanish for the critical value. The condition may be stated as follows:

THEOREM. *The function $f(x)$ has a maximum (or minimum) for $x = a$, if the first one of the derived functions $f'(x)$, $f''(x)$ . . . which does not vanish for $x = a$, is of even order and negative (or positive).*

We have from Taylor's expansion, written in finite form,

$$f(x + h) = f(x) + f'(x)\,h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots + \frac{f^n(x + \theta h)}{n!}h^n,$$

and

$$f(x - h) = f(x) - f'(x)\,h + \frac{f''(x)}{2!}h^2 - \frac{f'''(x)}{3!}h^3 + \dots + \frac{f^n(x + \theta h)}{n!}h^n.$$

Replacing x by a, we have, after transposing the first term of the second member of the identity,

$$f(a+h) - f(a) = f'(a)\,h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \dots + \frac{f^n(a+\theta h)}{n!}h^n, \qquad (1)$$

$$f(a-h) - f(a) = -f'(a)\,h + \frac{f''(a)}{2!}h^2 - \frac{f'''(a)}{3!}h^3 + \dots + (-1)^n\frac{f^n(a+\theta h)}{n!}h^n. \qquad (2)$$

If for $x = a$ the given function has a maximum, then $f(a)$ must exceed the value of the function for all values of the variable in the neighborhood of a; in other words, the left-hand member of both (1) and (2) must be negative for all values of h sufficiently small. The value of h can be taken so small that $hf'(a)$ will be numerically greater than the sum of the remaining terms of the second member. However, $f(a+h) - f(a)$ and $f(a-h) - f(a)$ cannot both be negative unless $f(a) = 0$ and $f''(a)$ is negative, assuming that $f''(a)$ does not vanish. It may, however, happen that both $f'(a)$ and $f''(a)$ vanish. In this case, in order to have both $f(a+h) - f(a)$ and $f(a-h) - f(a)$ negative, $f'''(a)$ must vanish and $f^{\text{IV}}(a)$ must be negative. In general, in order that $f(x)$ shall have a maximum value for $x = a$, the first derivative that does not vanish must be of even order and negative.

In order that $f(x)$ shall have a minimum for $x = a$, the two expressions $f(a+h) - f(a)$ and $f(a-h) - f(a)$ must be positive. This requires that the first derivative that does not vanish shall be even and positive. The argument is similar to that given above and is left to the student.

Ex. Examine the function $\cos x + \cosh x$ for maxima and minima.

We have

$$f(x) = \cos x + \cosh x,$$
$$f'(x) = -\sin x + \sinh x.$$

For $x = 0$, $f'(x) = 0$; hence $x = 0$ is a critical value. We have further

$$f''(x) = -\cos x + \cosh x, \quad \text{whence } f''(0) = 0,$$
$$f'''(x) = \sin x + \sinh x, \quad \text{whence } f'''(0) = 0,$$
$$f^{\text{IV}}(x) = \cos x + \cosh x, \quad \text{whence } f^{\text{IV}}(0) = 2.$$

Since the fourth derivative is positive, and is the first that does not vanish, it follows that $f(x)$ is a minimum for $x = 0$.

EXERCISES.

Examine for maxima and minima the following functions:

1. $\tan^2 x - 2\tan x$.
2. $\sinh x - \sin x$.
3. $\sin x\,(1 + \cos x)$.
4. $pe^{ax} + qe^{-ax}$.
5. $\cosh u - u^2$.
6. $x \sin x$.
7. $3\cos\theta + \tan^2\theta$.
8. $\cos x - \log\cos x$.
9. $x^{-m} e^{mx}$.
10. $e^x + e^{-x} + 2\cos x$.

140. Evaluation of indeterminate forms. Algebraic methods of evaluating certain indeterminate forms were shown in the examples of Art. 13. For the form $\frac{0}{0}$, to which all other forms may be reduced, the differential calculus furnishes a general method of evaluation, which is developed as follows:

Let the given function be of the form $\frac{f(x)}{\phi(x)}$, which reduces to the form $\frac{0}{0}$ for $x = a$. The value of the limit $\underset{x \doteq a}{L}\frac{f(x)}{\phi(x)}$ is required. Expanding each term of this fraction by Taylor's development in finite form, we have

$$\frac{f(x)}{\phi(x)} = \frac{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n[a+\theta_1(x-a)]}{n!}(x-a)^n}{\phi(a) + \phi'(a)(x-a) + \frac{\phi''(a)}{2!}(x-a)^2 + \ldots + \frac{\phi^n[a+\theta_2(x-a)]}{n!}(x-a)^n}; \quad (1)$$

By hypothesis, $f(a)$ and $\phi(a)$ are each equal to zero. The above relation therefore reduces to

$$\frac{f(x)}{\phi(x)} = \frac{f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n[a+\theta_1(x-a)]}{n!}(x-a)^n}{\phi'(a)(x-a) + \frac{\phi''(a)}{2!}(x-a)^2 + \ldots + \frac{\phi^n[a+\theta_2(x-a)]}{n!}(x-a)^n} \cdot \quad (2)$$

Dividing both terms of this fraction by $(x-a)$ and passing to the limit, we have

$$\underset{x \doteq a}{L} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}. \tag{3}$$

If $f'(a)=0$ and $\phi'(a) \neq 0$, this limit reduces to zero; if $f'(a) \neq 0$ and $\phi'(a)=0$, it becomes infinite.

If $f'(a)=0$ and $\phi'(a)=0$, the limiting value of the given function can be obtained by dividing the terms of the expanded form of the fraction by $(x-a)^2$ and then passing to the limit. The result is

$$\underset{x \doteq a}{L} \frac{f(x)}{\phi(x)} = \frac{f''(a)}{\phi''(a)}. \tag{4}$$

Similarly, if $f''(a)$ and $\phi''(a)$ are both zero, we divide by $(x-a)^3$ and again take the limit, and so on.

We may therefore state the general law of procedure as follows:

To evaluate the indeterminate form $\frac{0}{0}$, *differentiate the numerator and the denominator of the given fraction and substitute the critical value of the variable in the result.*

The function $\frac{f(x)}{\phi(x)}$ may also assume the indeterminate form $\frac{0}{0}$ when x becomes infinite. The limiting value may still be found by considering $\frac{f'(x)}{\phi'(x)}$; for, we have upon putting $x=\frac{1}{z}$,

$$\underset{x \doteq \infty}{L} \frac{f(x)}{\phi(x)} = \underset{z \doteq 0}{L} \frac{-f'\left(\frac{1}{z}\right)\frac{1}{z^2}}{-\phi'\left(\frac{1}{z}\right)\frac{1}{z^2}} = \underset{z \doteq 0}{L} \frac{f'\left(\frac{1}{z}\right)}{\phi'\left(\frac{1}{z}\right)} = \underset{x \doteq \infty}{L} \frac{f'(x)}{\phi'(x)}. \tag{5}$$

Form $\frac{\infty}{\infty}$. When the function $\frac{f(x)}{\phi(x)}$ takes the form $\frac{\infty}{\infty}$, it can be reduced to the form $\frac{0}{0}$, by writing it in the form $\frac{\frac{1}{\phi(x)}}{\frac{1}{f(x)}}$. It can then be evaluated according to the law just stated.

It may also be shown that if $\frac{f'(x)}{\phi'(x)}$ has a limit as x approaches a definite number or becomes infinite, then $\frac{f(x)}{\phi(x)}$ converges to the same limit.* This principle often affords a convenient method of evaluating this indeterminate form; for we need only to differentiate the numerator and the denominator and then pass to the required limit.

Form $\infty - \infty$. When a function takes the indeterminate form $\infty - \infty$ it may be reduced to the fundamental form $\frac{0}{0}$ by writing it as follows:

$$f(x) - \phi(x) = \frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{\phi(x)}} = \frac{\frac{1}{\phi(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)\cdot\phi(x)}}. \tag{6}$$

Often, however, a simpler transformation will reduce the function to one of the forms $\frac{0}{0}, \frac{\infty}{\infty}$. No general rule can be given, but that transformation should be selected which gives the simplest form.

Form $0 \times \infty$. When a function $f(x) \cdot \phi(x)$ takes the form $0 \times \infty$ for $x = a$, it may be reduced to the type $\frac{0}{0}$ by writing it in the form

$$\frac{f(x)}{\frac{1}{\phi(x)}}. \tag{7}$$

Forms 0^0, ∞^0, 1^∞. The indeterminate forms 0^0, ∞^0, 1^∞ arise from a function of the form $[f(x)]^{\phi(x)}$. This function may be reduced to the type form $\frac{0}{0}$ as follows:

Let $$y = [f(x)]^{\phi(x)},$$
whence $$\log y = \phi(x) \cdot \log\ [f(x)]. \tag{8}$$

Since, for each of the given forms, (8) takes the form $0 \times \infty$, the solution is effected by (7).

Ex. 1. Evaluate $\frac{\tan x - \sin x}{x^3}$, for $x = 0$.

We have

$$\frac{f'(x)}{\phi'(x)} = \frac{\sec^2 x - \cos x}{3x^2}\Big]_{x=0} = \frac{0}{0};$$

$$\frac{f''(x)}{\phi''(x)} = \frac{2\sec^2 x \tan x + \sin x}{6x}\Big]_{x=0} = \frac{0}{0};$$

$$\frac{f'''(x)}{\phi'''(x)} = \frac{4\sec^2 x\tan^2 x + 2\sec^4 x + \cos x}{6}\Big]_{x=0} = \frac{1}{2}.$$

Hence $$\underset{x=0}{L}\ \frac{\tan x - \sin x}{x^3} = \frac{1}{2}.$$

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* See Pierpont's *Theory of Functions*, Vol. I, p. 305. The special student of mathematics would do well to read Arts. 455-459 in the same volume.

Ex. 2. Evaluate $\sec \frac{1}{2}\pi x \cdot \log \frac{1}{x}$ for $x = 1$.

This function takes the form $0 \times \infty$; hence we write it in the form

$$\frac{\log \frac{1}{x}}{\frac{1}{\sec \frac{1}{2}\pi x}} = \frac{\log \frac{1}{x}}{\cos \frac{1}{2}\pi x},$$

which for $x = 1$ takes the form $\frac{0}{0}$. Differentiating the numerator and the denominator, we get

$$\left.\frac{\frac{d}{dx}\log \frac{1}{x}}{\frac{d}{dx}\cos \frac{1}{2}\pi x}\right]_{x=1} = \left.\frac{-\frac{1}{x}}{-\frac{1}{2}\pi \sin \frac{1}{2}\pi x}\right]_{x=1} = \frac{2}{\pi}.$$

Ex. 3. Evaluate $x^{\frac{1}{1-x}}$ for $x = 1$.

Let $y = x^{\frac{1}{1-x}}$; then $\log y = \frac{1}{1-x}\log x = \frac{\log x}{1-x}$, which has the form $\frac{0}{0}$. Hence by the general rule,

$$\underset{x \doteq 1}{L} \log y = \left.\frac{\frac{d}{dx}\log x}{\frac{d}{dx}(1-x)}\right]_{x=1} = \left.\frac{\frac{1}{x}}{-1}\right]_{x=1} = -1,$$

$$\text{whence} \quad \underset{x \doteq 1}{L} \quad y = \frac{1}{e}.$$

EXERCISES.

Evaluate the following indeterminate forms:

1. $\left.\dfrac{x^2 + 4x - 21}{x^3 - 3x^2 - 4x + 12}\right]_{x=3}.$

2. $\left.\dfrac{x^3 + 2x^2 - x - 2}{x^2 + 10x + 16}\right]_{x=-2}.$

3. $\left.\dfrac{1 - \cos x}{x^2}\right]_{x=0}.$

4. $\left.\dfrac{1}{x}(a^x - b^x)\right]_{x=0}.$

5. $\sec x - \tan x]_{x=\frac{\pi}{2}}.$

6. $\left.\dfrac{\log x}{\sqrt{1-x}}\right]_{x=1}.$

7. $\left.\dfrac{\sinh x - x}{\sin x - x}\right]_{x=0}.$

8. $\left.\dfrac{\tan \theta - \theta}{\theta - \sin \theta}\right]_{\theta=0}.$

9. $\left.\dfrac{x^x - x}{1 - x + \log x}\right]_{x=1}.$

10. $\left.\dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}\right]_{x=0}.$

11. $x\left(a^{\frac{1}{x}} - 1\right)\Big]_{x=\infty}.$

12. $\left.e^x \sin \dfrac{1}{x}\right]_{x=\infty}.$

13. $\left.\dfrac{2}{x} - \cot \dfrac{x}{2}\right]_{x=0}.$

14. $\left.\dfrac{\pi}{4x} - \dfrac{\pi}{2x\left(e^{\frac{\pi}{x}} + 1\right)}\right]_{x=0}.$

15. $\sin x \log \cot x]_{x=0}.$

16. $(\sin \theta)^{\tan \theta}]_{\theta=\frac{\pi}{2}}.$

17. $x^x\Big]_{x=0}$.

18. $(1+x^2)^{\frac{1}{x}}\Big]_{x=0}$.

19. $(e^x+x)^{\frac{1}{x}}\Big]_{x=0}$.

20. $\sec 2\,\theta \cos 5\,\theta\Big]_{\theta=\frac{\pi}{2}}$.

21. $\dfrac{\text{inv. tanh}\, x - x}{x^3}\Big]_{x=0}$.

22. $\dfrac{\arctan x - x}{x^3}\Big]_{x=0}$.

23 $\dfrac{e^x \sec x - 1}{x}\Big]_{x=0}$.

24. $\dfrac{\sec \frac{\pi x}{2}}{\log(1-x)}\Bigg]_{x=1}$.

25. $\dfrac{x}{\sin^3 x} - \dfrac{1}{\tan^2 x}\Big]_{x=0}$.

26. $\dfrac{x}{x-1} - \dfrac{1}{\log x}\Big]_{x=1}$.

MISCELLANEOUS EXERCISES.

1. Draw the graphs of the functions $1, 1-\frac{x^2}{2!}, 1-\frac{x^2}{2!}+\frac{x^4}{4!}$, respectively, and compare them with the graph of $\cos x$.

2. Test for convergence the following series, and determine the interval of convergence:

(a) $1+\frac{1}{6}\frac{x}{3}+\frac{1}{11}\frac{1\cdot 4}{1\cdot 2}\left(\frac{x}{3}\right)^2+\frac{1}{16}\frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\left(\frac{x}{3}\right)^3+\dots$

(b) $1+\frac{x}{e}+\frac{1}{2!}\frac{x^2}{e^2}+\frac{1}{3!}\frac{x^3}{e^3}+\dots$

(c) $1+x\cos a+\frac{x^2}{2!}\cos 2\,a+\frac{x^3}{3!}\cos 3\,a+\dots$

(d) $\cos x+\frac{1}{2}\cos 2\,x+\frac{1}{3}\cos 3\,x+\dots$

3. Expand the following functions:

(a) $\tan\theta$. (b) $e^x\cos x$. (c) $\cos^n x$. (d) $\dfrac{x}{e^x-1}$. (e) $\log\sec^2\dfrac{\theta}{2}$.

4. Show that $\displaystyle\int_0^x \frac{\sin x}{x}\,dx = x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}+\dots$

5. Derive a similar series for $\displaystyle\int_0^x \frac{\sinh x}{x}\,dx$.

6. Show that $(1+x)^{\frac{1}{x}} = e\left(1-\frac{1}{2}x+\frac{11}{24}x^2-\frac{7}{16}x^3+\dots\right)$.

Suggestion: Let $u=(1+x)^{\frac{1}{x}}$, whence $\log u = \dfrac{\log(1+x)}{x}$. Make use of the series already found for $\log(1+x)$ to determine the successive derivatives.

7. From the expansion for $\log(n+h)$ and $\log(n+1)$, derive the *approximate rule of proportional parts*, viz.:

$$\frac{\log(n+h)-\log n}{\log(n+1)-\log n}=\frac{h}{1}.$$

From this rule find log 7.523, knowing that log 7.52 = 2.0176 and log 7.53 = 2.0189.

8. Develop $f(5 \pm h)$ in powers of h, and determine the values of $f(x) = x^2(16 - x)$ for the following values of x: 4.7, 4.8, 4.9, 5, 5.1, 5.2, 5.3.

9. Show by development in series that $e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x$, and $e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x$.

10. Referring to Fig. 122, let b denote one half of chord C. Then the area of the circular segment between arc and chord is given by the formula

$$A = r^2 \text{ arc sin } \frac{b}{r} - b\sqrt{r^2 - b^2}.$$

From the expansions of arc sin $\frac{b}{r}$ and $\sqrt{r^2 - b^2}$, show that

$$A = \frac{2}{3}b^2\left[\frac{b}{r} + \frac{3}{10}\left(\frac{b}{r}\right)^3 + \frac{9}{56}\left(\frac{b}{r}\right)^5 + \frac{5}{48}\left(\frac{b}{r}\right)^7 + \dots\right].$$

11. Derive a series for the same area using the height h of the segment instead of the radius r.

12. Develop the function $y = \dfrac{\text{arc sin } x}{\sqrt{1 - x^2}}$ in series.

Suggestion: Multiply by $\sqrt{1 - x^2}$ and differentiate. The resulting equation is $(1 - x^2)\dfrac{dy}{dx} - xy = 1$. Now assume $y = A + Bx + Cx^2 + \dots$ and determine the coefficients.

13. Derive the approximate formula

$$\tan(\theta + m) = \tan\theta + m\sec^2\theta.$$

14. The strength of an electric current as shown by a tangent galvanometer is given by $i = C\tan\phi$, where C is a constant and ϕ is the deflection of the needle. Show that an error m in the reading of the angle gives a relative error of $\dfrac{2m}{\sin 2\phi}$ in the current.

15. Evaluate the following expressions by the use of series:

(a) $\left.\dfrac{e^x - 1}{x}\right]_{x=0}$; (b) $\left.\dfrac{x - \sin x}{x^3}\right]_{x=0}$; (c) $\left.\dfrac{e^x - e^{-x}}{\sin x}\right]_{x=0}$.

16. Calculate the following:

(a) Sin 10° and cos 10° to five figures.

(b) Logarithms (natural) of 17, 31, 61, correct to four places.

(c) $\sqrt[7]{2184}$.

17. Prove that the expansion of an *even* function of x, that is, one for which $f(x) = f(-x)$, will contain only even powers of x, while if $f(x) = -f(-x)$ the expansion will contain only odd powers of x. Illustrate by several functions.

18. Making use of the known series for e^x, $\log(1 + x)$, $\sin x$, etc., derive series for the following functions:

(a) $\dfrac{e^x}{1 + x}$. (b) $e^x \log(1 + x)$. (c) $\sqrt{1 \pm \sin 2x}$. (d) $\cos^2 x$.

19. By substituting mx for x in Ex. 9, prove *DeMoivre's theorem*, namely,

$$(\cos x + \sqrt{-1}\sin x)^m = \cos mx + \sqrt{-1}\sin mx.$$

20. Given the series

$$y = e\left(\sin x - \frac{\sin 2x}{2!} + \frac{\sin 3x}{3!} - \frac{\sin 4x}{4!} + \ldots\right)$$

(a) Test it for uniform convergence in the interval $(0, \pi)$.

(b) Can the series be integrated term by term? Show geometrically by plotting several of the approximation curves.

21. By means of the exponential series and the identity

$$e^{x\sqrt{-1}}\, e^{y\sqrt{-1}} = e^{x\sqrt{-1}+y\sqrt{-1}}$$

show that $2\sin x\cos x = \sin 2x$.

CHAPTER XVI

APPLICATIONS TO PLANE CURVES

141. Order of contact. In the preceding chapters, we have discussed many of the simpler and more direct applications of calculus to geometry. In the present chapter, we shall consider some further applications to plane curves which will enable the student to use the principles of the calculus to determine the character of a curve at certain critical points in order that its graph may be traced with greater ease and accuracy.

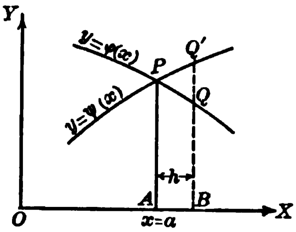

Fig. 123.

Let the equations of two given plane curves be

$$y = \phi(x) \text{ and } y = \psi(x). \qquad (1)$$

For some value of x, as $x = a$, suppose the curves to intersect, as at P, Fig. 123. Then for this value of x,

$$\phi(a) = \psi(a).$$

Now consider a second value of x, as $x = a + h$, where h is a small positive number. This value is represented in the figure by the abscissa OB. The corresponding ordinates BQ and BQ' are respectively

$$y = \phi(a + h), \quad y' = \psi(a + h).$$

Expanding by Taylor's formula, we have

$$\left.\begin{aligned} y &= \phi(a) + h\phi'(a) + \frac{h^2}{2!}\phi''(a) + \dots \\ y' &= \psi(a) + h\psi'(a) + \frac{h^2}{2!}\psi''(a) + \dots \end{aligned}\right\} \qquad (2)$$

The difference $y' - y$, represented by the intercept QQ', is therefore given by the expression

$$[\psi(a) - \phi(a)] + h[\psi'(a) - \phi'(a)] + \frac{h^2}{2!}[\psi''(a) - \phi''(x)] + \dots \qquad (3)$$

We may look upon the intercept QQ' as measuring in a way the closeness of contact of the curves. Hence we may distinguish the following cases:

(1) If the expression in the first brackets alone vanishes, the derivatives are unequal for $x = a$, and the curves merely intersect at that point.

(2) If in addition, the second term vanishes, that is, if $\psi'(a) = \phi'(a)$, and the succeeding derivatives are unequal, the curves have a common tangent at the point $x = a$, and are said to have *contact of the first order.* See Fig. 124.

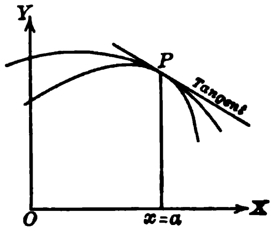

Fig. 124.

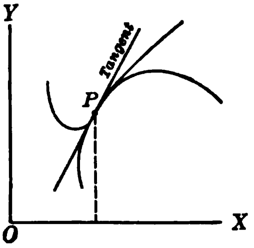

Fig. 125.

(3) If the first three terms all vanish, that is, if we have simultaneously

$$\psi(a) = \phi(a),$$
$$\psi'(a) = \phi'(a),$$
$$\psi''(a) = \phi''(a),$$

while the succeeding derivatives are unequal, the contact is of the *second order.* See Fig. 125.

Similarly, if $\psi(a) = \phi(a)$, and all the derivatives up to and including the nth are equal each to each, the curves are said to have *contact of the nth order.*

The highest order of contact that any curve can have with a given curve depends, in general, upon the number of arbitrary constants in the equation of that curve. This statement will be illustrated by the following special cases. Suppose, for example, that the curve $y = \phi(x)$ is the straight line

$$y = mx + b.$$

Then the successive derivatives are

$$\frac{dy}{dx} = m, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 0, \text{ etc.}$$

We are required to give this line the closest possible contact with the given curve $y = \psi(x)$ at the point $x = a, y = \psi(a)$; we must therefore satisfy as many as possible of the conditions

$$\phi(a) = \psi(a), \quad \phi'(a) = \psi'(a), \quad \phi''(a) = \psi''(a), \text{ etc.}$$

The first two conditions give the equations

$$ma + b = \psi(a), \quad m = \psi'(a),$$

and these are just sufficient to determine the constants m and b. The third condition cannot be fulfilled except at a point where $\psi''(a) = 0$, that is, at a point of inflexion in the curve $y = \psi(x)$. In general, therefore, a straight line cannot have contact of an order higher than the first with any given curve.

If we assume the curve $y = \phi(x)$ to be the circle

$$(x - a)^2 + (y - b)^2 = r^2,$$

there are three constants to be determined. Three equations are needed to determine them, and therefore three conditions will be fulfilled. Hence, in general, the circle cannot have a contact of order higher than the second with another curve.

It may be shown that: *If n is the number of arbitrary constants in the equation of the curve $y = \phi(x)$, this curve cannot have, in general, an order of contact higher than the $(n - 1)$st with any other curve $y = \psi(x)$.*

Since the equation of a parabola has four constants, the parabola may have at most a contact of the third order with another curve. The equation of a central conic has in general five constants; hence such a conic can have a contact of the fourth order with a curve whose equation contains at least as many constants. As in the case of a straight line, special points may be found where these curves will have contact of higher order with a given curve.

142. The osculating circle. When a curve has a contact of the highest order with a given curve, it is called the *osculating curve*. In particular, the circle that has contact of the highest order at a given point of a curve $y = \psi(x)$ is called the *osculating circle* at the given point.

The coördinates of the center and the radius of the osculating circle may be determined by the principles just discussed. Let the point (m, n) be the center of the circle, and let R denote the radius. Then if we denote by t and u the variable coördinates of the circle to distinguish them from the coördinates of the curve $y = \psi(x)$, we have for the equation of the circle,

$$(t-m)^2 + (u-n)^2 = R^2. \tag{1}$$

The conditions for a second-order contact, when $t = x$, are:

$$\left.\begin{aligned} u &= y \\ \frac{du}{dt} &= \frac{dy}{dx} \\ \frac{d^2u}{dt^2} &= \frac{d^2y}{dx^2} \end{aligned}\right\} \tag{2}$$

From (1) we have by successive differentiation,

$$(t - m) + (u - n)\frac{du}{dt} = 0, \tag{3}$$

$$1 + \left(\frac{du}{dt}\right)^2 + (u - n)\frac{d^2u}{dt^2} = 0. \tag{4}$$

Substituting in (3) and (4) the values given by (2), we obtain

$$(x - m) + (y - n)\frac{dy}{dx} = 0, \tag{5}$$

$$1 + \left(\frac{dy}{dx}\right)^2 + (y - n)\frac{d^2y}{dx^2} = 0, \tag{6}$$

whence by easy transformations we find

$$\left.\begin{aligned} y - n &= -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \\ x - m &= \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \end{aligned}\right\} \tag{7}$$

From these equations the coördinates m and n of the center may be found. For x and y are substituted the coördinates of the point of contact, and the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ are obtained from the equation of the curve, $y = \phi(x)$. Finally substituting in (1) from (7), we get for the radius,

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \tag{8}$$

From (8) it follows that the radius of the osculating circle is precisely the radius of curvature of the given curve at the point of contact (see Art. 73). Hence, since the circle of curvature and the osculating circle have the same radius and are both tangent to the given curve at the same point, they must coincide. In fact, the coördinates of the center, given by (7), are the same as derived in Art. 73 from another definition.

At the contact point, the osculating circle and the given curve have the same tangent, since by hypothesis the value of $\frac{dy}{dx}$ is the same for the circle and the curve. It follows that the radius of the circle drawn to the point of contact coincides with the normal to the curve. Furthermore, since the value of $\frac{d^2y}{dx^2}$ is the same for the curve and the osculating circle, the direction of bending is the same for both at the point of contact; that is, both circle and curve are either concave upward or concave downward at the contact point.

EXERCISES.

1. Find the equation of the straight line that has contact of the first order with the curve $4y = x^2 - 12$ at the point (6, 6).

2. Find the equation of the osculating circle at the same point.

3. Find a straight line that has second-order contact with the curve

$$y = x^3 - 6x^2 + 8x + 4.$$

4. Find the osculating circle at the point (3, 0) of the curve $y = \log(x - 2)$.

5. Find the osculating circles at the extremities of the major and minor axes of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Show that the contact is of the third order at these points.

143. Envelopes. The equation of a plane curve in general contains in addition to the variables x and y certain constants which serve to fix the size and position of the curve. For example, consider the equation

$$(y - a)^2 = m\,(x - m),$$

which represents a parabola. The constants a and m determine the position of the axis and vertex, and the length of the latus rectum. As indicated in Art. 1, either of these constants may be an arbitrary constant, that is, a constant to which different values may be assigned at pleasure. If in the equation just given m is kept fixed while a is given a series of values, each of these values determines a particular parabola. All the parabolas, however, have the same latus rectum, their vertices lie on the same line $x = m$, and their axes are parallel. See Fig. 126. If, on the other hand, m is given a series of values while a is kept fixed, there results a series of parabolas, all having their axes on the line $y = a$. Their vertices lie at different points on this line, and their latera recta are different. See Fig. 127. A series of curves thus obtained by varying an arbitrary constant is called a *system* or *family* of curves, and the constant is often called the *parameter* of this family.

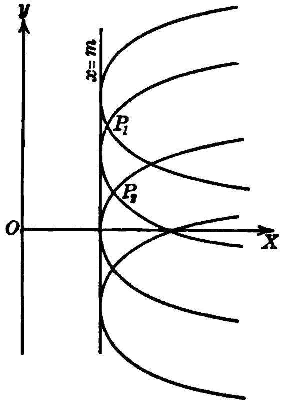

Fig. 126.

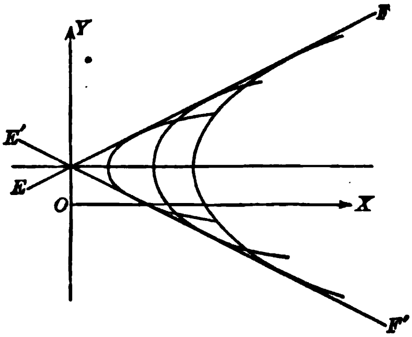

Fig. 127.

The curves of the system are of the same character; that is, they are all parabolas, circles, or spirals, as the case may be, and differ only in position, size, or form. The curves may or may not intersect. In the example just cited, they do intersect. A case in which they do not intersect is that of the family of circles given by the equation

$$x^2 + y^2 = r^2,$$

in which r is the parameter.

The presence of a parameter α in an equation is emphasized by writing the functional equation in the form

$$f(x, y, \alpha) = 0.$$

If the curves of a system intersect, the position of the point of intersection of two curves depends upon the relative position of the curves. As the curves approach coincidence, the intersection in general approaches a limiting position. The locus of these limiting points of intersection is called the *envelope* of the system. For example, in Fig. 126, the intersections P_1, P_2, etc., approach more and more nearly the line $x = m$ as the axes of the intersecting parabolas approach each other. Hence this line is the envelope of the system of parabolas.

The definition of an envelope suggests a method of determining its equation. The following example shows an application.

Ex. Required the envelope of the system of parabolas

$$(y - a)^2 = m(x - m),$$

derived by giving arbitrary values to the parameter m.

The equations of two neighboring curves are

$$(y - a)^2 = m(x - m), \tag{1}$$

and

$$(y - a)^2 = (m + h)[x - (m + h)], \tag{2}$$

where h is an arbitrary increment of m. The point of intersection of these curves is found by solving (1) and (2) as simultaneous equations. We thus obtain

$$x = 2m - h,$$
$$y = a \pm \sqrt{m(m - h)}. \tag{3}$$

The limiting position of this point when the intersecting curves approach coincidence is found by making h approach zero. Hence this limiting position is given by the coördinates

$$x = 2m,$$
$$y = a \pm m. \tag{4}$$

Eliminating m between these equations, we get

$$y = a \pm \frac{x}{2} \tag{5}$$

as the relation between the coördinates of the limiting position of the intersection of *any* two consecutive curves. Hence (5) is the equation of the locus of such points; that is, the equation of the envelope of the system of curves. Evidently the envelope consists of two straight lines, as shown in Fig. 127. Line EF has the equation $y = a + \frac{x}{2}$, line $E'F'$ the equation $y = a - \frac{x}{2}$.

144. Analytical determination of envelopes. We may develop the general method of determining an envelope in the following manner. Let two curves of the system have the equations

$$f(x, y, \alpha) = 0, \tag{1}$$

$$f(x, y, \alpha + h) = 0, \tag{2}$$

respectively. Using the theorem of mean value, Art. 32, we have

$$f(x, y, \alpha + h) - f(x, y, \alpha) = h f'_\alpha(x, y, \alpha + \theta h), 0 < \theta < 1, \tag{3}$$

whence by reason of (1) and (2)

$$f'_\alpha(x, y, \alpha + \theta h) = 0.$$

If now we make h approach zero, we have

$$f'_\alpha(x, y, \alpha) = \frac{\partial}{\partial \alpha} f(x, y, \alpha) = 0. \tag{4}$$

Equations (1) and (4) are the parametric equations of the envelope of the system of curves given by (1). When convenient α may be eliminated from (1) and (4), giving for the envelope a single equation in x and y.

Ex. 1. Find the envelope of the system of lines

$$y = mx + \frac{a}{m}, \tag{1}$$

obtained by assigning different values to m.

We have

$$f(x, y, m) = -y + mx + \frac{a}{m} = 0,$$

whence

$$\frac{\partial f}{\partial m} = x - \frac{a}{m^2} = 0. \tag{2}$$

From (2), $\frac{a}{m} = mx$. Substituting this value in (1), we have

$$y = 2mx \quad \text{or} \quad m = \frac{y}{2x},$$

and substituting this value of m in (2), we obtain finally

$$y^2 = 4ax.$$

In some cases the equation of the system of curves contains two parameters, between which, however, there exists a given relation. The following example illustrates the method usually adopted in finding the envelope.

Ex. 2. Find the envelope of a line of fixed length which moves with its extremities on two fixed lines at right angles to each other.

Let c denote the length of the line. Taking the coördinate axes as the fixed lines, the equation of the moving line is

$$\frac{x}{a} + \frac{y}{b} = 1. \qquad (1)$$

Also
$$a^2 + b^2 = c^2. \qquad (2)$$

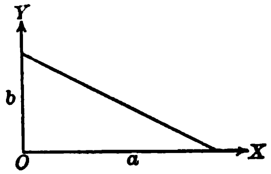

Fig. 128.

The lines constituting the system are obtained by varying both a and b subject to the relation expressed by equation (2). One of the parameters, say b, may be eliminated between (1) and (2), and then the envelope may be found by the usual method for one parameter. The following procedure, however, entails less labor. Equations (1) and (2) are differentiated with respect to a, b being regarded as a function of a. The following equations result:

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{db}{da} = 0, \qquad (3)$$

$$a + b\frac{db}{da} = 0. \qquad (4)$$

The elimination of $\frac{db}{da}$ between (3) and (4) gives the relation

$$b^3x = a^3y,$$

or
$$\frac{x}{a} = \frac{a^2}{b^3}y.$$

Then from (1),
$$\frac{a^2 + b^2}{b^3}y = 1,$$

or
$$c^2y = b^3.$$

Likewise
$$c^2x = a^3.$$

From (2)
$$c^2 = a^2 + b^2 = (c^2x)^{\frac{2}{3}} + (c^2y)^{\frac{2}{3}},$$

whence
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}},$$

which is the desired equation of the envelope.

EXERCISES.

1. Find the envelope of the system of circles given by the equation

$$(x - 3)^2 + (y - m)^2 = 25,$$

where m is an arbitrary constant.

2. A line so moves that the sum of its intercepts on the coördinate axes is constant. Find the envelope.

3. Find the envelope of the family of ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are parameters connected by the relation

$$a + b = \text{const.}$$

4. Find the envelope of the system of ellipses in Ex. 3, on the assumption that the area of the ellipse remains constant.

5. Find the envelope of the normals to the parabola $y^2 = 4ax$.

145. Roulettes and involutes. Suppose a plane curve, as AB, Fig. 129, to roll without slipping on a fixed curve MN; for this purpose, we may regard the curves as the boundaries of two disks. Points E, F on the rolling curve describe curves e, f on the fixed plane. Curves generated in this manner are called *roulettes*. Cycloids and trochoids are examples of roulettes in which the rolling curve is a circle. (See Art. 71.)

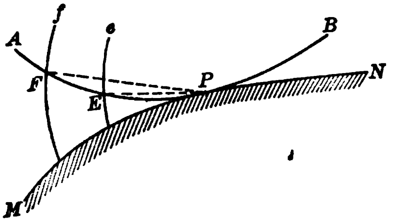

Fig. 129.

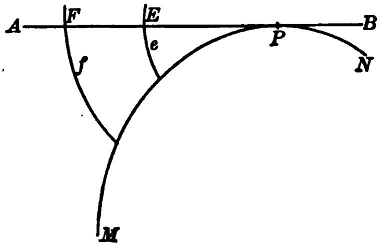

Fig. 130.

If the rolling curve is replaced by a straight line, Fig. 130, the roulettes are called *involutes* of the fixed curve; thus, curves e and f are involutes of the curve MN. Evidently a curve has an infinite number of involutes.

We may also consider the involute to be generated by a point on a flexible thread or cord wrapped around the fixed curve. As the cord is unwrapped, any point of it generates an involute.

Certain properties of involutes are evident from the manner in which they are generated.

1. *Any normal to an involute is a tangent of the fixed curve.* This property may be proved by purely mechanical principles. Referring to Fig. 130, the point of contact P, considered as a point of the rolling line AB, has at the instant of contact no motion in the direction of the line, since by hypothesis there is no slipping. Furthermore, since the curves remain in contact, P can have no motion perpendicular to AB. It follows that the point P of AB is at rest, and the curve as a whole is rotating about P as a center. Points E and F are therefore moving in directions perpendicular to the lines PE and PF respectively. Now the point E moves in the direction of the tangent to the curve it describes; hence the line PE is the normal to the curve e at E, and likewise PF is normal to curve f at F. In the case of the involute, Fig. 130, the normals PE and PF coincide with the rolling line AB, which is tangent to MN; hence the normal of the involute is tangent to the curve.

2. *Two involutes of the same curve intercept a constant distance on their common normal.* This follows at once from the manner in which the involute is generated. Because of this property, involutes are sometimes called parallel curves.

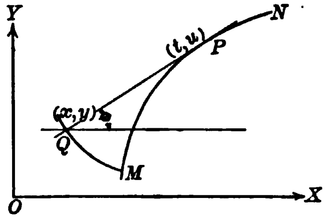

Fig. 131.

146. Curvature of involutes. Let PQ, Fig. 131, be one position of the rolling straight line, P being the point of contact, and Q a point on the involute. Let t, u denote the coördinates of the variable point P on the fixed curve MN, and x, y those of the point Q on the involute. From the geometry of the figure, we have at once

$$(t - x) \tan \phi = u - y. \tag{1}$$

Since QP is tangent to MN, we have $\tan \phi = \dfrac{du}{dt}$; but since QP is the normal to the involute at the point Q, we have also $\tan \phi = -\dfrac{dx}{dy}$. Substituting this value in (1) we obtain

$$(t - x) + (u - y) \frac{dy}{dx} = 0. \tag{2}$$

Differentiating (2) with respect to u, we have

$$\frac{dt}{du} - \frac{dx}{du} + \left(1 - \frac{dy}{du}\right)\frac{dy}{dx} + (u - y)\frac{d\left(\frac{dy}{dx}\right)}{du} = 0.$$

But since
$$\frac{dt}{du} = -\frac{dy}{dx},$$
this equation reduces to

$$-\frac{dx}{du} - \frac{dy}{du}\frac{dy}{dx} + (u - y)\frac{d\left(\frac{dy}{dx}\right)}{du} = 0.$$

Multiplying through by $\frac{du}{dx}$, we obtain finally,

$$-1 - \left(\frac{dy}{dx}\right)^2 + (u - y)\frac{d^2y}{dx^2} = 0,$$

whence

$$u = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \tag{3}$$

Substituting this value of u in (2), we have

$$t = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}. \tag{4}$$

Comparing (3) and (4) with (2) and (3) of Art. 73, it appears that the point P is the center of curvature and PQ is the radius of curvature at the point Q of the involute.

It is not true of roulettes in general that the point of contact of the rolling curves is the center of curvature of the roulette. Thus in Fig. 129, the center of curvature of e at the point E lies somewhere on EP or EP produced, but not at P. Only in the case of the involute does the center of curvature coincide with the center of rotation.

147. Evolute of a curve. From the preceding article it appears that a given curve is the locus of the centers of curvature of each of its involutes. When two curves C_1 and C_2 are so related that C_1 is an involute of C_2, we call C_2 the *evolute* of C_1. Curve C_2 may have other involutes than C_1, but C_1 has only the one evolute C_2.

Two properties of the evolute follow from the manner of describing the involute as a roulette.

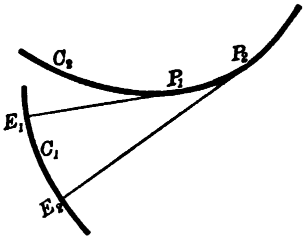

Fig. 132.

1. *Any normal to the curve C_1 is tangent to the evolute C_2.*

2. *The difference between two radii of curvature of a given curve is equal to the arc of the evolute between the points of contact of the radii with the evolute.*

Thus in Fig. 132, we have

$$P_2E_2 - P_1E_1 = \text{arc } P_1P_2.$$

To obtain the equation of the evolute of a given curve, we combine the equation of the curve

$$y = f(x) \tag{1}$$

with the equations

$$n = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}, \tag{2}$$

$$m = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}, \tag{3}$$

which give the coördinates m, n of the center of curvature. If x, y, and the derivatives be eliminated between these equations, the result will be a relation between m and n, the variable coördinates of the evolute. Various special expedients may be used in the elimination.

Ex. 1. Find the evolute of the rectangular hyperbola $xy = \frac{a^2}{2}$.

We have

$$\frac{dy}{dx} = -\frac{a^2}{2x^2}; \qquad \frac{d^2y}{dx^2} = \frac{a^2}{x^3}.$$

Hence

$$n = \frac{a^2}{2x} + \frac{4x^4 + a^4}{4a^2x} = \frac{4x^4 + 3a^4}{4a^2x},$$

$$m = x + \frac{4x^4 + a^4}{8x^3} = \frac{12x^4 + a^4}{8x^3}.$$

Therefore

$$m + n = \frac{8x^6 + 12x^4a^2 + 6x^2a^4 + a^6}{8a^3x^3}$$

$$= \left(\frac{2x^2 + a^2}{2ax}\right)^3,$$

and

$$m - n = \left(\frac{2x^2 - a^2}{2ax}\right)^3.$$

Finally

$$(m + n)^{\frac{2}{3}} - (m - n)^{\frac{2}{3}} = 2a^{\frac{2}{3}},$$

which is the desired equation of the evolute.

Ex. 2. Find the evolute of the cycloid

$$x = a\theta - a \sin \theta,$$
$$y = a - a \cos \theta.$$

We readily obtain the derivatives $\frac{dy}{dx} = \cot \frac{\theta}{2}$, $\frac{d^2y}{dx^2} = -\frac{1}{4a \sin^4 \frac{\theta}{2}}$.

Substituting these values in the equations for m and n, we get

$$m = a\theta + a \sin \theta,$$
$$n = -a + a \cos \theta,$$

which are the parametric equations of the evolute.

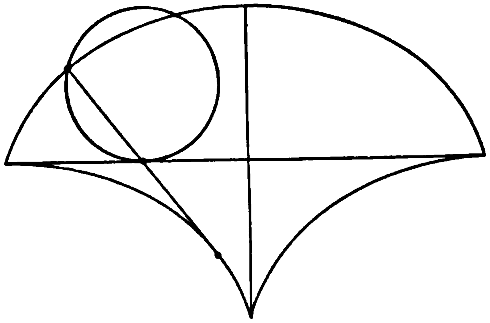

Fig. 133.

If now we transfer the origin to the point $(-\pi a, -2a)$ and denote by x and y the coördinates with reference to this new origin, we get

$$x = m + \pi a, \quad y = n + 2a,$$

whence

$$x = a(\theta + \pi) + a \sin \theta,$$
$$y = a + a \cos \theta.$$

If we denote $\theta + \pi$ by θ', we have

$$x = a\theta' - a \sin \theta',$$
$$y = a - a \cos \theta'.$$

Hence the evolute of the cycloid is an equal cycloid. The relative position of the two cycloids is shown in Fig. 133.

EXERCISES.

Find the evolutes of the following curves.

1. The parabola $y^2 = 4\,px$.

2. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

3. The circle $x^2 + y^2 = a^2$.

4. The astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

5. The tractrix $x = a \log \dfrac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$.

6. Show that the evolute of the parabola $y^2 = 4\,px$ is the envelope of the normals to the parabola.

148. Asymptotes, rectangular coördinates. The student has already met with curves having infinite branches, that is, portions of the curve that extend to infinity. The parabola and the hyperbola are illustrations of such curves. We shall now examine the tangents to such a branch. If the tangent still lies partly in the finite portion of the plane as the point of tangency recedes indefinitely, it is said to be asymptotic to the curve. An asymptote may, however, be more precisely defined as follows:

An *asymptote* to a curve is a straight line, lying partly within the finite region of the plane, which is the limiting position of the tangent as the point of tangency recedes indefinitely along an infinite branch.

As will be seen from this definition, we cannot speak of all tangents to an infinite branch of a curve as asymptotes, since not all such tangents lie partly within the finite portion of the plane. For example, the tangent to a parabola at infinity is not an asymptote, since it lies altogether at infinity. The hyperbola, on the other hand, has asymptotes, because the limiting position of the tangent to either infinite branch of the curve is a line lying partly within the finite region.

A tangent to a curve is obtained by regarding it as the limiting position of a secant line as the two points of intersection with the given curve are made to approach coincidence. The conditions for a tangent involve therefore finding multiple roots of the equation that results from combining the equation of the secant line with that of the curve. When the point of tangency recedes indefinitely, that is, when the tangent becomes an asymptote, this condition involves finding the condition that an equation shall have infinite roots.

Suppose, for example, we have given an equation of the nth degree,

$$a_0x^n + a_1x^{n-1} + \dots + a_n = 0. \tag{1}$$

The condition that some of the n roots of this equation shall become infinite is that the same number of coefficients of the highest powers of x shall vanish. This may be shown as follows. If we put $x = y^{-1}$, the given equation becomes

$$\frac{a_0}{y^n} + \frac{a_1}{y^{n-1}} + \dots + \frac{a_n}{1} = 0, \tag{2}$$

whose roots are also roots of the equation

$$a_0 + a_1y + \dots + a_ny^n = 0.$$

If now a_0 is allowed to approach zero, this equation has a root which also approaches zero, and consequently (1) has a root which becomes infinite. In the same way if a_0 and a_1 both vanish, the given equation has two infinite roots.

I. *Oblique asymptotes.* Let us apply this principle to the determination of asymptotes. Suppose the equation of the asymptote to be of the form

$$y = mx + b, \tag{3}$$

and let the equation of the given curve be

$$f(x, y) = 0, \tag{4}$$

where $f(x, y)$ is a rational integral function of x and y. If m is finite and different from zero, the asymptotes given by (3) will make oblique angles with the coördinate axes and hence are called *oblique asymptotes.*

In order that (3) shall be an asymptote to (4), the equation obtained by solving these equations simultaneously, namely,

$$f(x, mx + b) = 0, \tag{5}$$

must have two infinite roots. Consequently, the coefficients of the two highest powers of x in (5) must vanish. If we equate these coefficients of x to zero, we can then determine the values of m and b, and hence the asymptote desired.

Ex. 1. Given the curve whose equation is

$$x^4 - x^3y + xy - y^3 = 0,$$

to determine its asymptotes.

Substituting $y = mx + b$, the equation becomes

$$x^4(1 - m) + x^3(-m^3 - b) + \ldots - b^3 = 0.$$

From the coefficients of the two highest powers of x, we have

$$1 - m = 0,$$

$$m^3 + b = 0,$$

hence $m = 1$, and $b = -1$.

The asymptote therefore is

$$y = x - 1.$$

If in equation (5) the coefficients of x^n and x^{n-1} contain both m and b, the terms of degree lower than the $(n - 1)$st do not in general affect the determination of the asymptotes. However, if the coefficient of x^{n-1} is zero, or if the value of m obtained by equating the coefficient of x^n to zero is such as to cause that of x^{n-1} to vanish also, we must make use of the coefficient of the next lower degree in order to have two equations from which m and b can be uniquely determined. This coefficient, in general, will be of the second degree in b, and hence we have in this case two parallel asymptotes, since for each value of m there are then two values of b. If the coefficient of x^{n-2} is also zero, or vanishes for the value of m obtained from the coefficient of x^n we must solve the coefficient of x^n and x^{n-3} simultaneously in order to determine m and b. In this case we shall have, in general, three values of b for each value of m, and consequently three parallel asymptotes.

We may also obtain the oblique asymptotes to a curve by solving the equation of the given curve simultaneously with the equation

$$x = my + b,$$

and equating to zero the coefficients of the two highest powers of y.

II. *Asymptotes parallel to coördinate axes.* In case the asymptotes are parallel to one of the coördinate axes, m in the one or the other of the substitutions just discussed will be zero. We can, however, often determine such asymptotes more readily by inspection. To do so, we write the given equation in descending powers of x or of y. It will then have either of the following forms:

$$ax^n + (by + c)\,x^{n-1} + \ldots + t = 0, \tag{5}$$

$$\alpha y^n + (\beta x + \gamma)\,y^{n-1} + \ldots + \tau = 0. \tag{6}$$

If a and $by + c$ vanish, then two roots of (5) become infinite, and the line

$$by + c = 0 \tag{7}$$

is an asymptote. Similarly, if in (6) α and $\beta x + \gamma$ vanish, the line

$$\beta x + \gamma = 0 \tag{8}$$

is an asymptote to the given curve. Consequently, if the given equation when arranged in descending powers of x (or y) does not have the term x^n(or y^n), then the coefficient of x^{n-1}(or y^{n-1}) equated to zero gives an asymptote to the given curve.

If in (5), a, b, and c all vanish, then the coefficient of x^{n-2}, which is in general a quadratic in y, will, when equated to zero, determine two asymptotes, real or imaginary. A similar statement applies to (6).

Ex. **2**. The equation $4x^3 + 2x^2y - 6xy^2 - x^2 + 3y^2 - 1 = 0$ is a cubic in which y^3 is absent. Hence arranged in descending powers of y, it takes the form

$$0y^3 - (6x - 3)y^2 + (2x^2)y + (4x^3 - x^2 - 1) = 0.$$

If we make $6x - 3 = 0$, the coefficients of y^3 and y^2 both vanish. Hence the line $6x - 3 = 0$, or $x = \frac{1}{2}$ is an asymptote. For $x = \frac{1}{2}$, we have also $y = -\frac{3}{2}$, so that the asymptote cuts the curve at the point $(\frac{1}{2}, -\frac{3}{2})$.

Ex. **3**. The equation $x^2y^2 - 4xy^2 + 3x^2y - 4x^2 - 5 = 0$ is of the fourth degree, but lacks the terms in x^4, x^3, y^4, and y^3. It may be arranged in the forms

$$0x^4 + 0x^3 + (y^2 - 4)x^2 - (4y^2 - 3y)x - 5 = 0,$$
$$0y^4 + 0y^3 + (x^2 - 4x)y^2 + (3x)y - (4x^2 + 5) = 0.$$

From the first we have two asymptotes,

$$y + 2 = 0, \quad \text{and} \quad y - 2 = 0,$$

determined by placing the coefficients of x^2 equal to zero. From the second, the asymptotes $x = 0$, $x - 4 = 0$ are obtained.

The two methods given are sufficient to determine all the asymptotes to an algebraic curve. The first method will be found most convenient in determining asymptotes that make an oblique angle with the axes of coördinates, and the second in finding asymptotes parallel to the axes.

If the equation of the curve involves a transcendental function, the method suggested in Ex. 12, p. 376, will be found sufficient to determine the asymptotes.

EXERCISES.

Find the asymptotes of the following curves:

1. $y^3 = x^2(x - b)$.
2. $y^2 = x^2\left(\frac{x + a}{x - a}\right)$.
3. $y^2 = \frac{x^3}{2a - x}$.
4. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
5. $y^2(x - 2) = x^3$.
6. $y^3 - y^2 - x^2 + x^3 = 0$.
7. $x^2y^2 - c^2(x^2 + y^2) = 0$.
8. $x^3 + xy^2 - y^3 = 0$.
9. $x^3 - xy^2 + ay^2 = 0$.
10. $y = \frac{8a^3}{x^2 + 4a^2}$.
11. $y^3 = 6x^2 + x^3$.
12. $xy = 1$.
13. $x^3 - y^3 - x^2 + 2y^2 = 0$.
14. $xy^2 + x^2y + 2x^2 - 3xy - 2y^2 = 0$.
15. $x^2y^2 - x^3y + x^2 - 4y^2 = 0$.
16. $x^5 + y^5 - 5ax^3y = 0$.
17. $y^3 - x^2y + 2y^2 + 4y + x = 0$.
18. $x^4 - y^4 = a^2xy$.
19. $x^2y + xy^2 = 8$.
20. $y + xy - x^2 = 0$.
21. $x^2y^2 - x^3 - y^3 = 0$.

149. Asymptotes, polar coördinates. If in the equation of the curve $\rho = f(\theta)$ the radius ρ becomes infinite for some value of θ, denoted by θ_0, the curve has an asymptote, provided the tangent at an infinitely distant point passes within a finite distance of the pole. This is the case when, for $\theta = \theta_0$, the subtangent $\rho^2 \frac{d\theta}{d\rho}$ is finite.

The construction for the asymptote is shown in Fig. 134.

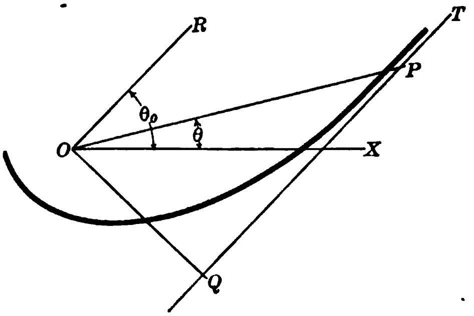

Fig. 134.

Through the pole O, the line OR is drawn, making angle $XOR = \theta_0$. From O perpendicular to OR, the subtangent $OQ = \rho^2 \left.\frac{d\theta}{d\rho}\right]_{\theta_0}$ is laid off. Then through Q the asymptote QT is drawn parallel to OR.

Let P be a point on the asymptote ; then from the triangle OPQ we obtain the relation

$$OP = \frac{OQ}{\sin OPQ} = \frac{OQ}{\sin(\theta_0 - \theta)}\,. \tag{1}$$

Denoting the subtangent of the asymptote by S, we have therefore for the polar equation of the asymptote,

$$\rho = \frac{S}{\sin(\theta_0 - \theta)}\,. \tag{2}$$

Ex. Find the asymptotes of the curve $\rho = a \tan \theta$.

For $\theta_0 = \frac{\pi}{2}$, $\rho = \infty$; also for $\theta_0 = \frac{3\pi}{2}$, $\rho = \infty$.

The subtangents for these values of θ_0 are

$$S = \frac{\rho^2 d\theta}{d\rho}\Bigg]_{\frac{\pi}{2}} = a \sin^2\theta\Bigg]_{\frac{\pi}{2}} = a,$$

and

$$S = a \sin^2\theta\Big]_{\frac{3}{2}\pi} = a.$$

Hence there are two asymptotes perpendicular to the initial line OX, each at a distance a from the pole. See Fig. 135.

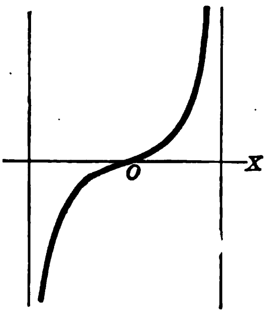

Fig. 135.

EXERCISES.

Find the asymptotes of the following curves. In each case sketch the curve and show the approximate position of the asymptotes.

1. The hyperbola $\rho = \frac{a(1 - e^2)}{c \cos\theta - 1}$.
2. The lituus $\rho = \frac{2\pi}{\sqrt{\theta}}$.
3. The hyperbolic spiral $\rho = \frac{2\pi}{\theta}$.
4. $\rho = a(\sec 2\theta + \tan 2\theta)$.
5. $\rho \cos 2\theta = a$.
6. $\rho - a = \frac{b}{\sin\theta}$.
7. $\rho = 2a \sin\theta \tan\theta$.
8 $\rho = \frac{a}{1 - \cos\theta}$.
9. $\rho = a\left(1 + \frac{1}{\cos 2\theta}\right)$.
10. $\rho = a \sec 2\theta$.

150. Singular points. In general, the derivative $\frac{dy}{dx}$ has at any point a single determinate value which measures the slope of the curve at that point. It may happen, however, that at certain points of the curve the derived function takes the indeterminate form $\frac{0}{0}$. Such points are called *singular points*, and the corresponding values of the variables x and y are called *singular values*. At a singular point the slope of the curve cannot be determined by the usual method, but it is possible to use the method of Art. 140 for the evaluation of indeterminate forms. An example will illustrate this statement.

Let the equation of the curve be

$$f(x, y) = x^3 - 3\,xy + y^3 = 0.$$

Then we have for the derived function

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} = \frac{3\,x^2 - 3\,y}{3\,x\ - 2\,y}.$$

Now for the point (0, 0), which is a point on the curve, the derived function takes the form $\frac{0}{0}$. To evaluate this form we take the x-derivative of both numerator and denominator in the usual way. Denoting by p the value of the derivative at the critical point, the resulting equation is

$$p = \left.\frac{6\,x - 3\,p}{3 - 2\,p}\right]_{0,0} = -\frac{3\,p}{3 - 2\,p}.$$

Solving this equation for p, we obtain

$$p = 0,\ p = 3.$$

Hence at the point (0, 0), the derivative has two distinct values.

We shall find that the evaluation of the derived function at a singular point always leads to at least a quadratic equation in p. If a second differentiation of numerator and denominator is necessary in order to evaluate this function, the resulting equation will be a cubic in p, and so on. Hence at a singular point there are two or more values of p, that is, of the derivative. Of these values, however, two or more may be equal, and some may be imaginary.

The geometrical interpretation of the multiplicity of values of p is simply that at a singular point, there are two or more tangents to the curve; in other words, the singular point is common to two or more branches of the curve.

Some of the singularities possessed by algebraic curves are shown in Fig. 136.

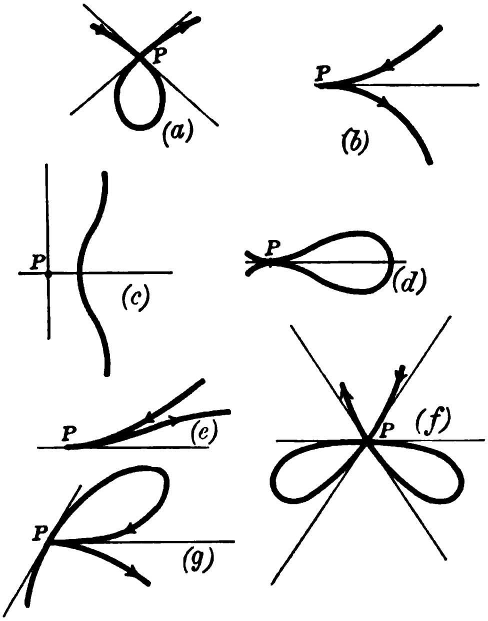

Fig. 136.

If p has two real and distinct values, there are two tangents, as at (*a*). The point of intersection P of the two branches is called a *double point* or *node.*

At (*f*) is shown a *triple point.* Here there are three distinct tangents corresponding to three real and unequal values of p. A triple point is equivalent to three nodes.

If the two values of p are real and equal, the two tangents are coincident. In this case two branches are tangent to each other at the singular point. If the branches pass through the point of contact, as shown at (*d*), the singular point is called a *tacnode.* If the two branches stop at the point, as shown at (*b*) and (*e*), the singular point is called a *cusp.* A cusp is said to be of the *first species* when the branches are on opposite sides of the tangent, as at (*b*), and it is of the *second species* when the branches lie on the same side, as at (*e*). We may regard a cusp as the limit of a node in which the loop has dwindled to a point, and the common tangents have become coincident.

If there are three values of p, of which two are equal, we may have a node and a cusp at the same point, as shown at (*g*).

Finally, p may have two values, both imaginary. In this case there are no real tangents, and therefore there are no real points of the curve in the neighborhood of the point in question. When this is the case, the singular point is called a *conjugate point*, as shown at (*c*).

151. Analytic condition for a singular point. Let the equation of the curve, written in the implicit form and without radicals, be

$$f(x, y) = 0. \tag{1}$$

Then for a singular point, we have

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = \frac{0}{0}; \tag{2}$$

that is, in addition to $f(x, y) = 0$, we must have $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$. Solving these equations simultaneously, we find the coördinates of points at which singularities occur.

To determine the character of the singularity we proceed as follows to deduce the necessary criteria. Evaluating the indeterminate form, and denoting as before the value of the derivative $\dfrac{dy}{dx}$ at the critical point by p, we have

$$p = -\frac{\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial x}\right)}{\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y}\right)}.$$

To perform the operation $\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial x}\right)$, let v denote $\dfrac{\partial f}{\partial x}$; then, since v is in general a function of x and y, we have

$$\frac{dv}{dx} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} p.$$

Replacing v by $\frac{\partial f}{\partial x}$, we have

$$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x\,\partial y}p.$$

By a similar process the expression for $\frac{d}{dx}\left(\frac{\partial f}{\partial y}\right)$ is obtained. The final result is

$$p = -\frac{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x\,\partial y}p}{\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y^2}p}. \tag{3}$$

Clearing (3) of fractions, we obtain a quadratic in p, viz.

$$\frac{\partial^2 f}{\partial y^2}p^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}p + \frac{\partial^2 f}{\partial x^2} = 0. \tag{4}$$

If in (4) the second partial derivatives vanish simultaneously for the singular values of x and y, then the expression (3) for p becomes indeterminate, and a second differentiation is required. Thus

$$\frac{dy}{dx} = p = -\frac{\frac{d}{dx}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x\,\partial y}p\right)}{\frac{d}{dx}\left(\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y^2}p\right)}$$

$$= -\frac{\frac{\partial^3 f}{\partial x^3} + 2\frac{\partial^3 f}{\partial x^2\partial y}p + \frac{\partial^3 f}{\partial x\,\partial y^2}p^2}{\frac{\partial^3 f}{\partial x^2\partial y} + 2\frac{\partial^3 f}{\partial x\,\partial y^2}p + \frac{\partial^3 f}{\partial y^3}p^2}, \tag{5}$$

whence

$$\frac{\partial^3 f}{\partial y^3}p^3 + 3\frac{\partial^3 f}{\partial x\,\partial y^2}p^2 + 3\frac{\partial^3 f}{\partial x^2\partial y}p + \frac{\partial^3 f}{\partial x^3} = 0. \tag{6}$$

If it should happen that the third partial derivatives all vanish, a third differentiation would be necessary, and the resulting equation in p would be a biquadratic; and so on.

Returning now to equation (4), let us suppose that the second partial derivatives are not all simultaneously zero. Then (4), as a quadratic in p, has two roots, and these give the two directions of the branches of the curve at the singular point.

Three cases are to be noted:

$$1.\quad \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\,\frac{\partial^2 f}{\partial y^2} > 0.$$

$$2.\quad \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\,\frac{\partial^2 f}{\partial y^2} = 0.$$

$$3.\quad \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\,\frac{\partial^2 f}{\partial y^2} < 0.$$

In the first case, the roots are real and unequal; hence the singular point is a node, Fig. 136 (*a*).

In the second case, the roots are real and equal, the two tangents are coincident, and the point is either a tacnode or a cusp.

In the third case, the roots are imaginary, and therefore there are no real tangents at the singular point. The point is therefore a conjugate point, Fig. 136 (*c*).

When the values of p are obtained from the cubic equation (5), they may all be real and distinct, in which case we have a triple point, Fig. 136 (*f*); two may be equal, as in Fig. 136 (*g*); or two may be imaginary and one real. In the last case there is but one branch of the curve passing through the singular point; hence, so far as appearance is concerned, there is nothing to distinguish this point from other points on the curve.

The following examples will illustrate various cases in the determination of singular points.

Ex. 1. Examine for singular points the curve

$$f(x, y) = 4x^3 - 12x^2 + 10x + xy + 11y - 3y^2 - 18 = 0.$$

We have

$$\frac{\partial f}{\partial x} = 12x^2 - 24x + 10 + y;$$

$$\frac{\partial f}{\partial y} = x + 11 - 6y.$$

The values $x = 1$, $y = 2$, satisfy the equations

$$f(x, y) = 0, \qquad \frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0;$$

hence (1, 2) is a singular point. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 24x - 24; \qquad \frac{\partial^2 f}{\partial x\,\partial y} = 1; \qquad \frac{\partial^2 f}{\partial y^2} = -6.$$

For $x = 1$, $\frac{\partial^2 f}{\partial x^2} = 0$; hence the equation for p is

$$-6p^2 + 2p = 0,$$

and the roots are $p = 0$ and $p = \frac{1}{3}$. The point (1, 2) is therefore a node, and the intersecting branches have the slopes 0 and $\frac{1}{3}$, respectively.

Ex. 2. Examine for singular points the curve

$$x^4 - 4x^2y + y^3 = 0.$$

We have

$$\frac{\partial f}{\partial x} = 4x^3 - 8xy, \qquad \frac{\partial f}{\partial y} = -4x^2 + 3y^2;$$

also

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 8y, \qquad \frac{\partial^2 f}{\partial x\,\partial y} = -8x, \qquad \frac{\partial^2 f}{\partial y^2} = 6y.$$

The coördinates of the origin (0, 0) satisfy the equations

$$f(x, y) = 0, \qquad \frac{\partial f}{\partial x} = 0, \qquad \text{and } \frac{\partial f}{\partial y} = 0.$$

Hence (0, 0) is a singular point. But for $x = 0$, $y = 0$, the second partial derivatives vanish, and in consequence the third partial derivatives must be used. For $x = 0$, we have

$$\frac{\partial^3 f}{\partial x^3} = 24x = 0; \qquad \frac{\partial^3 f}{\partial x^2\,\partial y} = -8; \qquad \frac{\partial^3 f}{\partial x\,\partial y^2} = 0; \qquad \frac{\partial^3 f}{\partial y^3} = 6.$$

From these, the cubic to determine the values of p is

$$6p^3 - 24p = 0,$$

whence $p = 0, 2$, and -2. Therefore the origin is a triple point and the slopes of the tangents to the branches are respectively $0, 2, -2$. The general form of the curve is shown in Fig. 137.

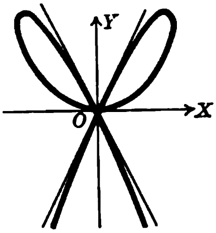

Fig. 137.

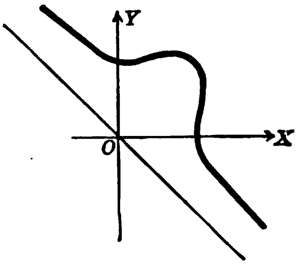

Fig. 138.

Ex. 3. Examine the curve

$$x^3 - 3x^2 - 3y^2 + y^3 = 0.$$

The partial derivatives are

$$\frac{\partial f}{\partial x} = 3x^2 - 6x;$$

$$\frac{\partial f}{\partial y} = 3y^2 - 6y;$$

$$\frac{\partial^2 f}{\partial x^2} = 6x - 6;$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = 0;$$

$$\frac{\partial^2 f}{\partial y^2} = 6y - 6.$$

Evidently the origin (0, 0) is a singular point. For $x = 0$, $y = 0$,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = -6,$$

and the equation giving the values of p is

$$-6p^2 - 6 = 0, \quad \text{or} \quad p^2 + 1 = 0,$$

whence $$p = \pm\sqrt{-1}.$$

Since the two values of p are imaginary, the origin is a conjugate point. The form of the curve is shown in Fig. 138.

Ex. 4. Examine the curve

$$ay = (x - b)^{\frac{5}{2}}.$$

The equation in the implicit form and freed from radicals is

$$a^2y^2 - (x - b)^5 = 0,$$

whence

$$\frac{\partial u}{\partial x} = -5(x - b)^4; \quad \frac{\partial u}{\partial y} = 2a^2y.$$

The values $x = b$, $y = 0$, satisfy the original equation $f(x, y) = 0$ and the equations

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0;$$

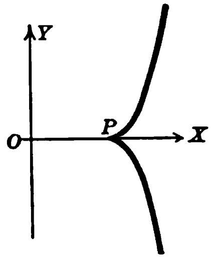

Fig. 139.

hence $(b, 0)$ is a singular point. At this point, we have

$$\frac{\partial^2 u}{\partial x^2} = -20(x - b)^3 = 0.$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = 0.$$

$$\frac{\partial^2 u}{\partial y^2} = 2a^2.$$

The quadratic equation is therefore

$$2a^2p^2 = 0,$$

whence $$p = 0.$$

For values of $x > b$, y has two values, equal but opposite in sign, but for $x < b$, y is imaginary. Since the two values of p are coincident and the curve lies wholly on one side of the point $(b, 0)$, that point is a cusp; and as the branches lie on opposite sides of the tangent, the cusp is of the first species

In tracing curves, it is of advantage not only to observe the general directions given in Art. 71, but also to find the asymptotes to the curve and to determine in advance the nature of its singular points. The student is now in position to trace much more complicated curves than those previously discussed.

EXERCISES.

Examine the following curves for singular points. In each case find the equations of the tangents at each singular point and trace the curve.

1. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.
2. $y^2 = x^3 - 4x^2$.
3. $cy^2 = x^3$.
4. $a^4y^2 = a^2x^4 - x^6$.
5. $x^4 - 2axy^2 + 2ay^2 = 0$.
6. $x^2 - 4x + 6y^3 = 0$.
7. $x^3 - y^3 - x^2 + 2y^2 = 0$.
8. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
9. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.
10. $y^2 = \frac{x^3}{2a - x}$.
11. $y^2 - x^3 - 2ax^2 - a^2x = 0$.
12. $y^2 = x(x + b)^2$.
13. $y = x^2 - x^{\frac{5}{2}}$.
14. $x^3 + 2y^3 + xy^2 - x^2y = 0$.
15. $y^2(x^2 - a^2) = x^4$.

MISCELLANEOUS EXERCISES.

1. Find the radius and coördinates of the center of the osculating circle in each of the following cases:

(a) $y^2 = 8x + 1$, at $(6, 7)$.
(b) $y = 3x^3 - 8x + 4$, at $(-2, -4)$.
(c) $y = \cos x$, at $(0, 1)$.
(d) $y = e^x$, at $(0, 1)$.

2. Find the parabola $y = Ax^2 + Bx + C$ which has the same osculating circle as the curve $y = x^3 - 6x^2 + 4x + 9$, at the point $(2, 1)$.

3. Show that the radius of curvature at any point of a cycloid is double the normal at the same point. (See Fig. 133.)

4. Prove that when two curves have contact of even order they cross each other; and that when the contact is of odd order the curves do not cross each other. Illustrate by examples.

5. If the equation of a curve can be written in the form

$$y = \pm (ax + b) + \phi(x),$$

where each term of $\phi(x)$ has some higher power of x in the denominator than in the numerator, show that the lines $y = \pm(ax + b)$ are asymptotes to the curve. In this way determine the asymptotes to the curve $x^3 - xy^2 + ay^2 = 0$.

6. Trace the following curves: .

(a) $y = (x^2 - 1)^2$.
(b) $a^4y^2 = a^2x^4 - x^6$.
(c) $y^2 = \frac{x^2(x + 1)}{x - 1}$.
(d) $x^4 - 3xy^2 + 2y^3 = 0$.

7. Trace the curves:

(a) $\rho = a \sec \theta \pm b$.
(b) $\rho = a(\cos \theta + \cos 2\theta)$.
(c) $\rho = b(\tan \theta - 1)$.
(d) $\rho = \frac{a \cos 2\theta}{\cos \theta}$.

8. Show that the ordinates of the curve $x^3 = xy + 1 = 0$ and of the parabola $y = x^2$ approach the same value as x increases. For this reason, the parabola is said to be a *curvilinear asymptote* to the curve.

9. Trace the curve $y + xy - x^3 = 0$. Find its rectilinear and curvilinear asymptotes.

10. Show that at the point (0, 0) the curve $y = \dfrac{x}{1 + e^{\frac{1}{x}}}$ has two terminating branches with different tangents.

11. Examine for singular points the following curves:

$$\text{(a) } x(x - a)^2 + y^2(x - 2a) = 0.$$
$$\text{(b) } (x^2 + y^2)^2 = 4x^2 + y^2.$$
$$\text{(c) } x^4 - 3xy + 6y^2 - y^4 = 0.$$

12. Find the asymptotes to the curves

$$\text{(a) } y^2 = 4x^2 + x^3; \qquad \text{(b) } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

by the following method: Take the values of the intercepts of the axes by the tangent to the curve at the point (x', y') when $x' = \infty$ and $y' = \infty$. If both intercepts are finite, or if one is finite, the tangent is an asymptote.

13. Show that if the degree of a curve is lower than the fourth the curve cannot have a cusp of the second species.

14. Deduce the following general method of determining the tangents at a multiple point (m, n) of a curve: Transform the origin to (m, n), arrange the resulting equation in ascending powers of x and y, and put the group of terms of lowest degree equal to zero.

15. Apply the method of Ex. 14 to find the tangents at the origin to the lemniscate $y^2 = x^2 - x^4$. By the same method find the tangents to the curve $(y - b)^2 = x(x - a)^2$ at the point (a, b).

16. Discuss the curve $p = ab^{\frac{\tau}{m + n\tau}}$ which represents De la Roche's equation for the pressure p of water vapor at the temperature. Find the point of inflexion, the point of discontinuity, and the asymptote. Trace the curve.

17. In Art. 24, Ex. 2, van der Waals' equation was mentioned. Give a complete discussion of this equation. Show the condition required for points of inflexion, and find these points for $T = 300°$. Show the general form of the graph for large and small values of T, using the values of the constants on p. 60.

18. Derive from van der Waals' equation the conditions that pv shall be a minimum, taking T constant.

19. Trace the "probability curve" $y = ke^{-a^2x^2}$, in which k and a are constants. Determine the value of x for which the curve has a maximum point, also the point of inflexion.

20. Given the witch whose equation is $y = \dfrac{8}{x^2 + 4}$. (a) Find the centroid of the area bounded by the curve, the X-axis, and the ordinates $x = -2$, $x = +3$. (b) Find the volume obtained by revolving the same portion of the curve about the X-axis.

21. Find the radius of curvature of the cardioid $\rho = 2(1 + \cos\theta)$ for $\theta = 30°$.

22. Prove that the envelope of the normals to a curve $y = f(x)$ is the evolute of the curve.

23. Prove that the evolute of the logarithmic spiral $\rho = ae^{k\theta}$ is a similar logarithmic spiral.

24. Find the equation of the evolute of the cissoid

$$y^2 = \frac{x^3}{2a - x}.$$

25. Find the envelope of the lines $x \cos \alpha + y \sin \alpha = p$, α being the variable parameter.

26. A circle of radius b rolls on a fixed circle of radius a, $a > b$, and a point on the rolling circle generates an epicycloid. Show (a) that the equations of the curve are

$$x = (a + b) \cos \phi - b \cos \frac{a + b}{b} \phi,$$

$$y = (a + b) \sin \phi - b \sin \frac{a + b}{b} \phi;$$

(b) that the radius of curvature is $\frac{4b(a + b)}{a + 2b} \sin \frac{a\phi}{2b}$;

(c) that the length of the curve is $\frac{8(a + b)b}{a}$;

(d) that the area between the curve and fixed circle is $\frac{\pi b^2 (3a + 2b)}{a}$.

CHAPTER XVII

APPROXIMATE INTEGRATION

152. Approximate integration by series. If a given integral $\int f(x)\,dx$ cannot be evaluated by the ordinary exact methods of integration, it is sometimes possible to develop $f(x)$ into an infinite series and integrate term by term. By taking a sufficient number of terms of the series resulting from the integration, we may approximate the given integral to any desired degree of accuracy. Of course, some such restriction as uniform convergence must be observed; however, the series arising in practice are for the most part power series, and, as we have seen, they converge uniformly.

Even when the function can be integrated directly it is sometimes convenient to use the method just described, for the series may be more easily handled in the subsequent operations, than a complicated integral. Examples of integration of series have already been given, Art. 137. The following examples give further illustrations.

Ex. 1. The perimeter of an ellipse, which has a for semi-major axis and e for eccentricity, is given by the integral

$$4\,a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2 \phi}\, d\phi.$$

This integral can be evaluated approximately by expansion in series. Thus we have

$$\sqrt{1 - e^2 \sin^2 \phi} = 1 - \frac{1}{2}\, e^2 \sin^2 \phi - \frac{1}{2} \cdot \frac{1}{4}\, e^4 \sin^4 \phi - \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6}\, e^6 \sin^6 \phi - \ldots$$

which is a power series. Integrating term-by-term, we have

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2 \phi}\, d\phi = \int_0^{\frac{\pi}{2}} d\phi - \frac{1}{2}\, e^2 \int_0^{\frac{\pi}{2}} \sin^2 \phi\, d\phi - \frac{1}{2} \cdot \frac{1}{4}\, e^4 \int_0^{\frac{\pi}{2}} \sin^4 \phi\, d\phi - \ldots$$

Evaluating these integrals separately, we get for the required perimeter the expression

$$2\,a\pi \left[1 - \left(\frac{1}{2}\right)^2 \frac{e^2}{1} - \left(\frac{1}{2} \cdot \frac{3}{4}\right)^2 \frac{e^4}{3} - \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}\right)^2 \frac{e^6}{5} - \ldots \right]$$

Ex. 2. Evaluate the integral

$$\int \frac{y}{y+c}\, dA,$$

applied to the area of a circle of radius a, Fig. 140.

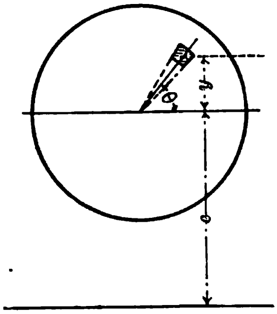

Fig. 140.

Passing to polar coördinates, we have

$$\Delta A = \rho\, \Delta\theta \Delta\rho, \qquad y = \rho \sin\theta,$$

whence the integral becomes

$$2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^a \frac{\rho^2 \sin\theta}{\rho \sin\theta + c}\, d\rho\, d\theta.$$

First integrating with respect to ρ, we have

$$\int_0^a \frac{\rho^2 \sin\theta}{\rho\sin\theta + c}\, d\rho = \int_0^a \rho\, d\rho - \frac{c}{\sin\theta} \int_0^a d\rho + \frac{c^2}{\sin\theta} \int_0^a \frac{d\rho}{\rho\sin\theta + c}$$

$$= \frac{a^2}{2} - \frac{ac}{\sin\theta} + \frac{c^2}{\sin^2\theta} \log\left(\frac{a}{c}\sin\theta + 1\right).$$

To simplify this second member, let us expand the third term in a series. We have as a result

$$\frac{a^2}{2} - \frac{ac}{\sin\theta} + \frac{c^2}{\sin^2\theta}\left[\frac{a}{c}\sin\theta - \frac{1}{2}\left(\frac{a}{c}\sin\theta\right)^2 + \frac{1}{3}\left(\frac{a}{c}\sin\theta\right)^3 - \dots\right],$$

which reduces to

$$a^2\left[\frac{a}{3c}\sin\theta + \frac{a^3}{5c^3}\sin^3\theta + \frac{a^5}{7c^5}\sin^5\theta + \dots\right]$$

$$- a^2\left[\frac{a^2}{4c^2}\sin^2\theta + \frac{a^4}{6c^4}\sin^4\theta + \frac{a^6}{8c^6}\sin^6\theta + \dots\right],$$

that is, the difference of two series, one in ascending odd powers of $\frac{a}{c}\sin\theta$, the other in even powers of the same argument. Denoting these series by S_1 and S_2, respectively, our original integral becomes

$$2a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S_1\, d\theta - 2a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S_2\, d\theta.$$

Between the given limits the integral of each term of the series S_1 reduces to zero. Hence the integral reduces to

$$-2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S_2\,d\theta = -2a^2\left[\frac{1}{4}\frac{a^2}{c^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\,d\theta + \frac{1}{6}\frac{a^4}{c^4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4\theta\,d\theta\right.$$

$$\left. + \frac{1}{8}\frac{a^6}{c^6}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^6\theta\,d\theta + \ldots\right]$$

Making use of the formula

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\pi,$$

we get, after slight reduction, the following series:

$$-\pi a^2\left[\frac{1}{4}\left(\frac{a}{c}\right)^2 + \frac{1}{8}\left(\frac{a}{c}\right)^4 + \frac{5}{64}\left(\frac{a}{c}\right)^6 + \frac{7}{128}\left(\frac{a}{c}\right)^8 + \ldots\right]$$

EXERCISES.

1. Find $\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{1-x^2}}$ to five figures.

2. Evaluate $\int_0^1\frac{dx}{\sqrt{1-x^4}}$.

3. Find the perimeter of the lemniscate $\rho^2 = a^2\cos 2\phi$.

4. The time of oscillation of a pendulum of length L is given by the expression

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

Integrate in series, and derive an approximate expression for T when k is small.

5. Expand $\int\frac{dx}{\sqrt{\sin x}}$ in series.

6. Evaluate $\int_0^1 e^{-x^2}\,dx$ by expansion in series. This integral is of fundamental importance in the theory of probability.

7. Show that

$$\int_0^{\frac{\pi}{2}}\frac{d\phi}{(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{3}{2}}} = \frac{\pi}{2\,ab^2}\left[1 - \tfrac{1}{4}k^2 - \tfrac{3}{64}k^4 - \tfrac{5}{256}k^6 - \ldots\right],$$

where $k^2 = 1 - \frac{b^2}{a^2}$.

8. With $k^2 = 1 - \frac{b^2}{a^2}$, show that

$$\int_0^{\frac{\pi}{2}} (b^2 \sin^2 \phi + a^2 \cos^2 \phi)^{\frac{3}{2}} d\phi = \frac{\pi a^3}{2}[1 - \tfrac{3}{4} k^2 + \tfrac{9}{64} k^4 + \tfrac{5}{256} k^6 + \ldots].$$

153. Approximate determination of areas. It is frequently necessary in engineering calculations to find the area of a figure whose bounding curves are more or less irregular, or at least have equations which are unknown. An example is shown in the determination of the horsepower of a steam engine from the area of the indicator diagram. If the diagram is bounded by known curves, the area can be readily found by the calculus methods previously explained. In the actual diagram, however, the equations of the bounding curves are unknown or can be obtained only with difficulty; hence the determination of the area by integration is not practicable.

As stated in Art. 82, the definite integral $\int_a^b f(x)\,dx$ is represented graphically by the area between the curve $y = f(x)$, the X-axis, and the ordinates corresponding to $x = a$ and $x = b$. Ordinarily, the definite integral is evaluated by means of the anti-derivative of $f(x)$. If, however, it is impossible or inconvenient to find the anti-derivative, the definite integral may be evaluated by the following method:

The curve $y = f(x)$ is plotted from $x = a$ to $x = b$, and the area between the curve, the X-axis, and the end-ordinates is determined approximately by one of the methods described in the following articles. The numerical measure of the area gives approximately the value of $\int_a^b f(x)dx$. The following example illustrates this method.

Ex. 1. Find the second moment of a triangle about an axis through a vertex parallel to the opposite side; see Fig. 141.

Take the given axis as the Y-axis, choose some point on it as an origin, and draw an axis OX. Taking a narrow strip parallel to the axis OY and at a distance x from it, the area of the strip is approximately

$$\Delta A = \frac{b}{h} x \,\Delta x,$$

and its second moment about OY is

$$\Delta I = x^2 \Delta A = \frac{b}{h} x^3 \Delta x.$$

Taking O as the origin, the curve OH whose equation is

$$y = \frac{b}{h}x^3$$

is drawn. The area OHH' under this curve gives the required second moment of the triangle about OY.

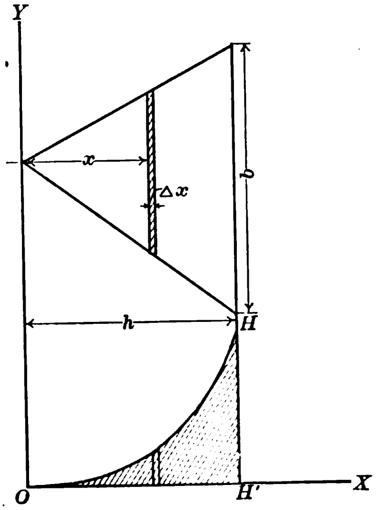

Fig. 141.

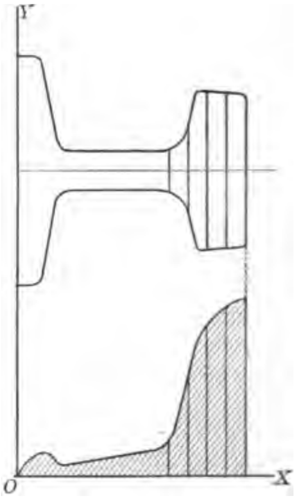

Fig. 142.

In this example, the result might have been obtained directly by integration, and the graphical solution is of no advantage. If, however, the area whose second moment is required has an irregular form, as in Fig. 142, the graphical solution is the only one that is practicable.

154. Simpson's rules. The problem of finding any plane area reduces to the problem of finding the area between a plane curve, an assumed X-axis, and two end-ordinates. Thus in Fig. 143 the area of the closed figure $AMBNA$ is found by subtracting the area between the curve ANB and OX from the area between AMB and OX.

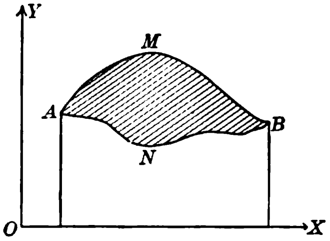

Fig. 143.

Let *ACE*, Fig. 144, be a part of a curve, and suppose the area between it and the line *NR* is required, the end-ordinates being *AN* and *ER*. Let the distance *NR* be divided into a number of equal parts, each equal to h, and through the points of division let ordinates y_1, y_2, y_3, etc., be drawn. Consider now the three points of the curve, *A*, *B*, and *C*. Taking *OB* as the Y-axis and *O* as the origin, the coördinates of these points are

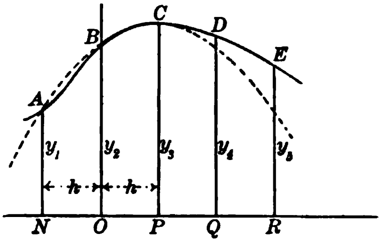

Fig. 144.

$$A \equiv (-h, y_1),\ \ B \equiv (0, y_2),\ \ C \equiv (h, y_3). \tag{1}$$

We now assume that the equation of the part of the curve *ABC* is

$$y = a_0 + a_1x + a_2x^2. \tag{2}$$

This is equivalent to the substitution of an arc of a parabola with a vertical axis for the actual curve. With this assumption, the area *NACP* is

$$\int_{-h}^{h} y\,dx = \int_{-h}^{h} (a_0 + a_1x + a_2x^2)\,dx = \frac{h}{3}(6\,a_0 + 2\,a_2h^2). \tag{3}$$

To determine the coefficients a_0 and a_2, substitute the coördinates given by (1) in (2). We thus obtain

$$\left.\begin{aligned} y_1 &= a_0 - a_1h + a_2h^2,\\ y_2 &= a_0,\\ y_3 &= a_0 + a_1h + a_2h^2.\end{aligned}\right\} \tag{4}$$

Solving for a_0 and a_2, we get

$$a_0 = y_2, \quad a_2 = \frac{y_3 + y_1 - 2\,y_2}{2\,h^2},$$

and substituting these values in (3), we have finally

$$\text{Area } NABCP = \frac{h}{3}(y_1 + 4\,y_2 + y_3). \tag{5}$$

If now we assume the part of the curve *CDE* replaced by a second parabolic arc, we get in the same way,

$$\text{Area } PCER = \frac{h}{3}(y_3 + 4\,y_4 + y_5),$$

and for the total area, we have

$$\text{Area } NAER = \frac{h}{3}(y_1 + 4\,y_2 + 2\,y_3 + 4\,y_4 + y_5).$$

This process may be repeated indefinitely; hence taking an odd number of ordinates $n + 1$, dividing the figure into n strips of equal width h, the approximate area is given by the formula

$$A = \frac{h}{3}(y_1 + 4\,y_2 + 2\,y_3 + \cdots + 2\,y_{n-1} + 4\,y_n + y_{n+1}). \tag{6}$$

This result gives the following rule, known as *Simpson's one-third* rule:

Take the sum of the end-ordinates, twice the sum of the intervening odd ordinates, and four times the sum of the even ordinates. Multiply the aggregate by one-third of the common distance between the ordinates.

If we take four ordinates y_1, y_2, y_3, y_4, including three spaces, we may pass through their ends A, B, C, and D, a third-degree parabola

$$y = a_0 + a_1x + a_2x^2 + a_3x^3.$$

Proceeding as before to determine the coefficients, we obtain

$$\text{Area } NADQ = \frac{3h}{8}(y_1 + 3\,y_2 + 3y_3 + y_4). \tag{7}$$

If therefore we divide the whole base into some number of parts n divisible by 3, and take the space in groups of three, or the ordinates in groups of four, we shall have for the successive partial areas,

$$A_1 = \frac{3h}{8}(y_1 + 3\,y_2 + 3\,y_3 + y_4),$$

$$A_2 = \frac{3h}{8}(\qquad\qquad y_4 + 3\,y_5 + 3\,y_6 + y_7),$$

$$A_3 = \frac{3h}{8}(\qquad\qquad\qquad\qquad y_7 + 3\,y_8 + 3\,y_9 + y_{10})$$

. . . .

Adding, we get for the total area,

$$A = \frac{3h}{8}(y_1 + 3\,y_2 + 3\,y_3 + 2\,y_4 + 3\,y_5 + \cdots + 3\,y_n + y_{n+1}). \tag{8}$$

This formula expresses *Simpson's three-eighths rule.*

Other rules of similar character may be obtained by taking five, six, or more ordinates in groups. Thus for a group of five ordinates,

$$A_1 = \frac{2h}{45}(7y_1 + 32y_2 + 12y_3 + 32y_4 + 7y_5).$$

A simple, though rather inaccurate, formula is obtained by taking the ordinates in groups of two. The curve joining the ends has then the equation

$$y = a_0 + a_1x,$$

and is therefore a straight line. Thus in Fig. 144, we join the points A, B, C, D, E, etc., by straight lines, and the actual figure is thus replaced by a series of trapezoids whose areas are respectively

$$A_1 = \frac{h}{2}(y_1 + y_2);\ A_2 = \frac{h}{2}(y_2 + y_3);\ A_3 = \frac{h}{2}(y_3 + y_4);\ \text{etc.}$$

The total area is therefore

$$A = \frac{h}{2}(y_1 + 2y_2 + 2y_3 + 2y_4 + \ldots + 2y_n + y_{n+1}). \qquad (9)$$

This formula is called the *trapezoidal formula.*

Durand's formula. From a discussion of the rules already given, Professor Durand has deduced the following rule, which combines some of the advantages of Simpson's rules and the trapezoidal formula.* It is as follows:

$$A = h(0.4y_1 + 1.1y_2 + y_3 + \ldots + y_{n-1} + 1.1y_n + 0.4y_{n+1}). \qquad (10)$$

In formulas (9) and (10) the number of intervals may be either even or odd.

Simpson's and Durand's rules may be used for the approximate determination of the volumes of solids. Let some line of the solid be taken as the X-axis, and let the solid be cut by equidistant planes perpendicular to this axis. Then if $y_1, y_2, \ldots y_n$ denote respectively the areas of the plane sections, the expression

$$\underset{\Delta x \doteq 0}{L} \sum y\,\Delta x = \int y\,dx,$$

taken between proper limits, gives the volume of the solid. As we have just shown, this integral is determined approximately by any one of the four rules (6), (8), (9), and (10).

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* See *Engineering News* for Jan. 18, 1894.

Ex. 1. Find by the approximate rules the area under the equilateral hyperbola $xy = 84$, from $x = 2$ to $x = 8$.

Taking unit intervals, we have for $x = 2, 3, \ldots, 8$, $y = 42, 28, 21, 16.8, 14, 12, 10.5$. By Simpson's one-third rule,

$$A = \tfrac{1}{3}\,[42 + 10.5 + 2\,(21 + 14) + 4\,(28 + 16.8 + 12)] = 116.67.$$

By the three-eighths rule,

$$A = \tfrac{3}{8}\,[42 + 10.5 + 2 \times 16.8 + 3\,(28 + 21 + 14 + 12)] = 116.67.$$

By Durand's rule,

$$A = 0.4\,(42 + 10.5) + 1.1\,(28 + 12) + 21 + 16.8 + 14 = 116.9.$$

By the trapezoidal rule,

$$A = 0.5\,(42 + 10.5) + 28 + 21 + 16.8 + 14 + 12 = 118.05.$$

The exact area is $\int_2^8 y\,dx = 84 \int_2^8 \frac{dx}{x} = 84 \log \frac{8}{2} = 116.45.$

Ex. 2. Find approximately the value of π from the formula

$$\frac{\pi}{4} = \text{arc tan } 1 = \int_0^1 \frac{dx}{1 + x^2}.$$

Here $y = \dfrac{1}{1 + x^2}$, and dividing the interval (0, 1) into 10 parts, whence $h = 0.1$, we get for the successive ordinates,

$$\begin{array}{lll}
y_1 = 1, & y_5 = .8620690 & y_9 = .6097561 \\
y_2 = .9900990 & y_6 = .8000000 & y_{10} = .5524862 \\
y_3 = .9615385 & y_7 = .7352941 & y_{11} = .5000000 \\
y_4 = .9174312 & y_8 = .6711409 &
\end{array}$$

Using Durand's rule, we get

$$\frac{\pi}{4} = .78540735, \text{ whence } \pi = 3.141629.$$

By Simpson's one-third rule, we get

$$\frac{\pi}{4} = .78539815, \text{ whence } \pi = 3.141593.$$

The latter result is correct to seven figures. The result by Durand's rule is too large by about one part in 87,000. It will be found that the trapezoidal rule gives a result too small by about one part in 2000.

Ex. 3. Find by Simpson's rule the volume of a sphere of radius a.

Dividing the diameter into four intervals, we have $h = \dfrac{a}{2}$, and $y = 0$, $y = \frac{3}{4}\pi a^2$, $y = \pi a^2$, $y = \pi a^2$, $y = \frac{3}{4}\pi a^2$, $y = 0$. Hence by Simpson's first rule,

$$\text{Volume} = \tfrac{1}{3} \cdot \frac{a}{2}[0 + 4\,(\tfrac{3}{4}\pi a^2 + \tfrac{3}{4}\pi a^2) + 2\,\pi a^2] = \tfrac{4}{3}\pi a^3.$$

EXERCISES.

1. Find $\int^{10} x^3\,dx$ by each of the four rules. Take $h = 1$.

2. Find log 2 approximately from the formula $\log 2 = \int_0^1 \frac{dx}{1+x}$. Take 10 intervals and use both the one-third and Durand's rule.

3. By Durand's rule, find $\int_0^{\frac{\pi}{3}} \sin\theta\,d\theta$. Take $h = 10° = \frac{\pi}{18}$.

4. Air at a pressure of 40 pounds per square inch expands from a volume of 6 cubic feet to a volume of 20 cubic feet. The expansion follows the law $pv = C$. Find approximately by Simpson's rule the work done during the expansion. Remember that the units must be consistent.

5. Find by Simpson's rule the volume of the frustum of a cone whose base radii are r_1 and r_2 and whose altitude is h.

6. Show that the area under the curve $y = kx^2$ from $x = 0$ to $x = a$ represents the volume of a cone of altitude a.

7. Find the curve whose area represents the first moment of a rectangle about one edge, Fig. 145. Find the area under the curve and thus obtain a value for the moment.

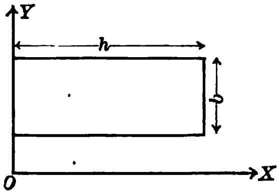

Fig. 145.

8. Conditions the same as in Ex. 7, but the second moment of the rectangle is required.

MISCELLANEOUS EXERCISES.

1. Find by Simpson's rules

(a) $\int_1^3 \frac{dx}{1+x^3}$; (b) $\int_0^6 x^{\frac{1}{2}}\,dx$; (c) $\int_2^7 \frac{dx}{x}$;

choosing proper divisions of the interval.

2. Devise a method of evaluating the important integral $\int \frac{y}{y+c}\,dA$ (Art. 152, Ex. 2) by the measurement of an area.

3. Fig. 142 shows a section of a steel rail. Copy this figure, divide it into strips parallel to the base, and by the aid of Simpson's or Durand's rule find approximately: (a) the area of the figure; (b) the first moment with respect to OY; (c) the second moment with respect to OY; (d) the position of the centroid.

4. In Fig. 140 replace the circle by a square of side a and find the value of the integral (a) expressed in finite terms (b) as an infinite series.

5. The formula giving under ideal conditions the flow of water through a circular orifice of radius a, Fig. 146, is

$$Q = 2\int_{-a}^{+a} \sqrt{a^2 - y^2}\,\sqrt{2\,g\,(h - y)}\,dy.$$

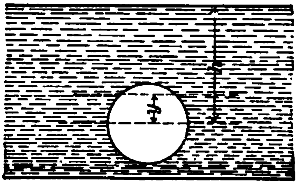

Fig. 146.

Expand the factors of the integrand and obtain the integral in the form of a series.

CHAPTER XVIII

ORDINARY DIFFERENTIAL EQUATIONS

155. Definitions. A differential equation is an equation involving derivatives or differentials, and usually the variables associated with the derivatives or differentials.

The following are examples of differential equations:

$$\frac{d^2y}{dx^2} + y = \cot x. \tag{1}$$

$$x^2\left(\frac{dy}{dx}\right)^2 - 2\left(xy + \frac{dy}{dx}\right) + y^2 = 0. \tag{2}$$

$$\frac{d^3y}{dx^3} = x^2 - 5. \tag{3}$$

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + z = 0. \tag{4}$$

$$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x\,\partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0. \tag{5}$$

An *ordinary differential equation* is one that involves only one independent variable and the derivatives with respect to that variable.

A *partial differential equation* is one that involves two or more independent variables and the corresponding partial derivatives. In the preceding examples (1), (2), and (3) are ordinary differential equations, (4) and (5) partial differential equations.

The *order* of a differential equation is that of the highest derivative involved. Thus equations (2) and (4) are of the first order, (1) and (5) of the second order, and (3) of the third order.

The *degree* of a differential equation is the degree of the derivative of highest order after the equation is cleared of fractions and radicals. Thus equation (2) is of the second degree, and the others are of the first degree.

156. Derivation and interpretation of differential equations. Before considering the solution of a differential equation it is advisable to examine the process by which a differential equation is obtained.

Suppose we have given an equation in the implicit form

$$f(x, y, c_1) = 0, \tag{1}$$

in which c_1 denotes an arbitrary constant. Differentiating (1) we obtain

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{2}$$

If now the constant c_1 be eliminated between (1) and (2), the result is a differential equation of the first order free from this constant.

For example, take the equation

$$(y - c_1)^2 = 4\,px. \tag{3}$$

The result of differentiation is

$$-4\,p + 2\,(y - c_1)\frac{dy}{dx} = 0; \tag{4}$$

and the elimination of c_1 between (3) and (4) gives the equation

$$x\left(\frac{dy}{dx}\right) = p. \tag{5}$$

If now we consider p an arbitrary constant, we can by a second differentiation get an equation free from p; thus

$$\left(\frac{dy}{dx}\right)^2 + 2\,x\frac{dy}{dx}\frac{d^2y}{dx^2} = 0. \tag{6}$$

The geometrical interpretation of this process of forming a differential equation is instructive. Equation (3) represents a parabola with its vertex on the Y-axis and its axis coincident with the line $y = c_1$. Equation (5), in which c_1 does not appear, represents the family of parabolas having their vertices on the Y-axis. These parabolas are, however, equal, having equal focal ordinates. Now by freeing the equation of p, we remove the restriction that the parabolas must be equal; hence equation (6) represents all parabolas having their vertices on the Y-axis.

As a second example take the equation

$$s = c_1 \cos \omega t + c_2 \sin \omega t. \qquad (7)$$

By successive differentiation we obtain

$$\frac{ds}{dt} = -\omega c_1 \sin \omega t + \omega c_2 \cos \omega t. \qquad (8)$$

$$\frac{d^2s}{dt^2} = -\omega^2 c_1 \cos \omega t - \omega^2 c_2 \cos \omega t. \qquad (9)$$

Combining (7) and (9), we have

$$\frac{d^2s}{dt^2} = -\omega^2 s, \qquad (10)$$

an equation free from the constants c_1 and c_2. Equation (10) expresses the property of the group of harmonic motions which may be obtained by varying the constants c_1 and c_2 in (7); namely, the property that the acceleration $\frac{d^2s}{dt^2}$ varies as the distance s from the origin.

EXERCISES.

1. From each of the following equations derive differential equations free from constants, and interpret geometrically. $\frac{d^2y}{dx^2}x^2 - 2\frac{dy}{dx}x + 2y = 0$

(a) $y = mx + b$; (b) $y = c_1x^2 + c_2x$; (c) $y = c_1 \sin x$. $y = y\,cdx$

$\frac{d^2y}{dx^2} = 0$

2. From the equation $s = \frac{1}{2}t^2 + v_0t + c$, derive a differential equation free from v_0 and c, and give a physical interpretation. $s' = 1$

3. Derive differential equations from the equations:

$\frac{d^2y}{dx^2} = m^2y$ (a) $y = c_1e^{mx} + c_2e^{-mx}$; (b) $y = e^{mx}(c_1 + c_2x)$. $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y = 0$

4. Find the differential equation which represents all parabolas having their axes coincident with the X-axis. $x^2 - \frac{dy}{dx}x + y = 0$

157. Solutions of a differential equation. The *solution* of a differential equation is a relation between the variables, not involving the derivatives, which will satisfy the given equation.

Three kinds of solutions may be distinguished: general, particular, and singular.

We have seen that in the process of deriving a differential equation from a given equation each differentiation resulted in the elimination of one constant. If the original equation has n constants, n differentiations produce n new equations, which with the original equation make a system of $n + 1$ equations, from which the n constants may be eliminated. The differential equation that results will involve derivatives of the nth order; hence the equation will be of the nth order. Now if this process is reversed and from the differential equation of nth order, the original equation with

n arbitrary constants is derived, that equation is called the *general* or *complete solution* of the differential equation.

A solution which is derived from the general solution by giving fixed values to some or all of the arbitrary constants is called a *particular solution.*

A solution without arbitrary constants, which cannot be derived from the general solution by assigning fixed values to the constants, is called a *singular solution.* This class of solutions will receive no attention in this chapter.

Consider, for example, the differential equation

$$\frac{d^2s}{dt^2} = -\omega^2 s,$$

derived in the preceding article. The general solution is

$$s = c_1 \cos \omega t + c_2 \sin \omega t,$$

containing two arbitrary constants. The following are particular solutions:

$$s = c_1 \cos \omega t \quad (c_2 = 0),$$
$$s = 2 \sin \omega t \quad (c_1 = 0, c_2 = 2),$$
$$s = 3 \cos \omega t - 4 \sin \omega t \quad (c_1 = 3, c_2 = -4).$$

A general solution may be obtained in several forms, but the constants involved must be related. For example, the differential equation $\frac{d^2s}{dt^2} = -\omega^2 s$ has the general solution

$$s = a \sin (\omega t + \epsilon),$$

as may be verified by substitution. This, however, is seen to be identical with the former solution

$$s = c_1 \cos \omega t + c_2 \sin \omega t,$$

if $$c_1 = a \sin \epsilon, \qquad c_2 = a \cos \epsilon.$$

A differential equation is considered as solved when a relation between the variables is expressed either in known functions or by means of integrals, even if the integrals cannot be evaluated by known methods.

EQUATIONS OF FIRST ORDER AND FIRST DEGREE.

158. Equations with variables separable. The form

$$M + N\frac{dy}{dx} = 0, \tag{1}$$

or the equivalent form

$$M\,dx + N\,dy = 0, \tag{2}$$

in which M and N are constants or functions of x and y, includes all differential equations of the first order and first degree.

When equation (2) can be written in the form

$$f(x)\,dx + F(y)\,dy = 0, \tag{3}$$

the solution may be obtained by direct integration. Thus we have

$$\int f(x)\,dx + \int F(y)\,dy = c. \tag{4}$$

In many cases, the variables may be separated by inspection.

Ex. 1. $$\cos x \sin^2 y - \sin^2 x \frac{dy}{dx} = 0.$$

By division we obtain readily

$$\frac{\cos x\,dx}{\sin^2 x} - \frac{dy}{\sin^2 y} = 0,$$

whence $$\int \frac{\cos x}{\sin^2 x}\,dx - \int \csc^2 y\,dy = 0,$$

or $$-\frac{1}{\sin x} + \cot y = c.$$

Ex. 2. If between two bodies there exists a repulsive force varying inversely as the square of the distance, as between two balls bearing electric charges of the same sign, the equation of motion is

$$\frac{v\,dv}{ds} = \frac{k}{s^2},$$

where k is a constant, and v the velocity.

Separating the variables, we have

$$v\,dv = k\frac{ds}{s^2},$$

whence by integration

$$v^2 = -\frac{2k}{s} + c.$$

If s_0 denotes the distance at the beginning of the motion, that is, when $v = 0$, $c = \frac{2k}{s_0}$, and the equation becomes

$$v^2 = 2k\left[\frac{1}{s_0} - \frac{1}{s}\right].$$

EXERCISES.

1. Find the solutions of the following equations:

(a) $\frac{dy}{dx} = \frac{x^2}{y^3}$.

(b) $(a - x)\,dy = (b - y)\,dx$.

(c) $y\sqrt{1 - x^2}\,dy = dx$.

(d) $x^2 \frac{dy}{dx} + y = 2$.

(e) $p\,dv + v\,dp = 0$.

(f) $\sec x \cos^2 y\,dx - \cos x \sin y\,dy = 0$.

2. Find the curve whose subtangent is n times the abscissa.

3. Find the curve whose slope is equal to the ordinate.

4. The general equation of the adiabatic change of state of air is

$$kp\,dv + v\,dp = 0.$$

Show that the equation of the adiabatic curve is $pv^k = \text{const}$.

5. If a particle moves under the action of an attractive force varying inversely as the square of the distance, the equation of motion is

$$v\frac{dv}{dx} = -\frac{k}{x^2}.$$

Integrate this equation and determine the constant of integration from proper initial conditions.

159. Homogeneous equations. If in the equation

$$M\,dx + N\,dy$$

M and N are homogeneous functions of x and y of the same degree, the substitution $y = vx$ reduces the equation to a form which permits separation of the variables.

Ex. $x^3 - 2\,xy^2 + 3\,x^2y\,\frac{dy}{dx} = 0.$

Let $y = vx$, whence $dy = v\,dx + x\,dv$. Writing the equation in the differential form and substituting these values for y and dy, we obtain

$$(x^3 - 2\,x^3v^2)\,dx + 3\,x^3v\,(v\,dx + x\,dv) = 0.$$

This equation after slight reduction takes the form

$$(1 + v^2)\,dx + 3\,vx\,dv = 0,$$

or

$$\frac{dx}{x} + \frac{3\,v\,dv}{1 + v^2} = 0.$$

Integrating, we have

$$\log x + \log\,(1 + v^2)^{\frac{3}{2}} = \log c$$

whence

$$x\,(1 + v^2)^{\frac{3}{2}} = c.$$

Replacing v by $\frac{y}{x}$, we get as the final solution,

$$\frac{1}{x^3}\,(x^2 + y^2)^{\frac{3}{2}} = c.$$

EXERCISES.

1. $(y - x)\dfrac{dy}{dx} + 2y = 0.$
2. $(x^2 + y^2)\,dx - 3\,xy\,dy = 0.$
3. $x^2y\,dx - (x^3 + y^3)\,dy = 0.$
4. $(xy + x^2)\,dy + y^2\,dx = 0.$
5. $(y^2 - xy) + x^2\dfrac{dy}{dx} = 0.$

160. Exact differential equations. The equation

$$M\,dx + N\,dy = 0 \tag{1}$$

is said to be exact when $M\,dx + N\,dy$ is the differential of some function u of x and y. In this case, the coefficients M and N are proportional to $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ respectively. See Art. 110.

As we have seen, the necessary condition that $M\,dx + N\,dy$ shall be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{2}$$

This condition is also sufficient; that is, if (2) is satisfied, equation (1) is exact. The proof is as follows:

Let z be the result of integrating $M\,dx$ with respect to x, that is, with y considered constant. We then have

$$\int^{x} M\,dx = z, \qquad M = \frac{\partial z}{\partial x}, \tag{3}$$

whence, under the conditions stated in the theorem of Art. 104,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}. \tag{4}$$

f, therefore, (2) is satisfied, we have

$$\frac{\partial N}{\partial x} = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right), \tag{5}$$

whence

$$N = \frac{\partial z}{\partial y} + F(y). \tag{6}$$

Here $F(y)$ plays the part of a constant of integration. See Art. 105. Substituting in (1) the expressions for M and N given in (3) and (6), we obtain

$$\frac{\partial z}{\partial x}\,dx + \frac{\partial z}{dy}\,dy + F(y)\,dy,$$

or

$$dz + F(y)\,dy = 0, \qquad \text{[Art. 107.]} \qquad (7)$$

which is integrable. Consequently the left-hand member of (1) may be replaced by *du*, and the condition (2) is also sufficient.

The solution of the given differential equation is derived at once from (7). It is

$$z + \int F(y)\,dy = c. \qquad (8)$$

Now from (6) we have

$$F(y)\,dy = \left(N - \frac{\partial z}{\partial y}\right)dy.$$

From (6) it is evident that the factor $N - \frac{\partial z}{\partial y}$ includes only those terms in N that do not contain x. We have therefore for general purposes the following simple rule:

Integrate $M\,dx$ considering y a constant, then integrate the terms in $N\,dy$ which do not contain x, and place the sum of the two integrals equal to a constant.

Ex. $(3x^2 + 2y^2)\,dx + y(4x - 9y)\,dy = 0.$

Since

$$\frac{\partial}{\partial y}(3x^2 + 2y^2) = \frac{\partial}{\partial x}(4xy - 9) = 4y,$$

the equation is exact. We have then

$$\int M\,dx = \int 3(x^2 + 2y^2)\,dx = x^3 + 2xy^2.$$

The term of N that does not contain x is $-9y^2$, and

$$\int 9y^2\,dy = 3y^3;$$

hence the solution is

$$x^3 + 2xy^2 - 3y^3 = c.$$

An equation that is not exact can sometimes be made exact by the introduction of an *integrating factor*. Thus the equation

$$kp\,dv + v\,dp = 0$$

becomes exact when multiplied by v^{k-1}; and the solution is

$$pv^k = c.$$

EXERCISES.

1. $x\,dy + y\,dx = 0.$

2. $\frac{y}{x}\,dx + \log x\,dy = 0.$

3. $e^y\,dx + xe^y\cdot dy = 0$

4. $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0.$

5. $(y + e^x \sin y)\,dx + (x + e^x \cos y)\,dy = 0.$

6. $(3\,x^2 + 4\,xy - y^2)\,dx + (2\,x^2 - 2\,xy + 3\,y^2)\,dy = 0.$

7. Find by inspection integrating factors for the following equations and obtain the solutions.

(a) $x\,dy - y\,dx = 0.$

(b) $2\,y\,dx + 5\,x\,dy = 0.$

(c) $(y - x)\,dy + y\,dx = 0.$

161. Linear equations of the first order. A differential equation is said to be linear when the degree of the dependent variable and its derivatives is not higher than the first. The linear equation of the first order has therefore the form

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

in which P and Q are functions of x, or constants.

The solution of equation (1) is obtained as follows: Let both members of (1) be multiplied by $e^{\int Pdx}dx$. The result is

$$e^{\int Pdx}dy + Py\,e^{\int Pdx}dx = Qe^{\int Pdx}dx. \tag{2}$$

The second member of (2) is integrable, since Q is a function of x alone. The first member is an exact differential, since it fulfills the necessary condition, namely,

$$\frac{\partial}{\partial x}\left(e^{\int Pdx}\right) = \frac{\partial}{\partial y}\left(Pye^{\int Pdx}\right) = Pe^{\int Pdx},$$

and its integral is

$$ye^{\int Pdx}.$$

Hence the solution of (1) is

$$y\,e^{\int Pdx} = \int Qe^{\int Pdx}dx + c. \tag{3}$$

It is seen that $e^{\int Pdx}$ is an integrating factor that renders the equation exact.

Ex. 1. $\dfrac{dy}{dx} - \dfrac{y}{1+x} = x^2.$

We have here
$$P = -\frac{1}{1+x},$$

whence
$$\int Pdx = -\int \frac{dx}{1+x} = \log \frac{1}{1+x}$$

and
$$e^{\int Pdx} = e^{\log \frac{1}{1+x}} = \frac{1}{1+x}.$$

From (3), we have therefore the solution
$$\frac{y}{1+x} = \int \frac{x^2}{1+x}\, dx$$

or
$$\frac{y}{1+x} = \frac{1}{2}x^2 - x + \log(1+x) + c.$$

Ex. 2. Find the solution of the equation
$$\frac{di}{dt} + \frac{R}{L}\, i = \frac{E}{L},$$

which under certain conditions gives the relation between electromotive force and current. See Art. 61 (d).

We have
$$e^{\int Pdt} = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}.$$

From (3) we obtain therefore
$$ie^{\frac{Rt}{L}} = \frac{E}{L}\int e^{\frac{Rt}{L}}\, dt,$$

whence
$$ie^{\frac{Rt}{L}} = \frac{E}{R}\, e^{\frac{Rt}{L}} + c,$$

or
$$i = \frac{E}{R} + ce^{-\frac{Rt}{L}}.$$

If, as in Art. 61, $i = 0$ when $t = 0$ then $c = -\dfrac{E}{R}$, and the equation takes the final form
$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right).$$

EXERCISES.

1. $\dfrac{dy}{dx} + \dfrac{xy}{1+x^2} = 1.$

2. $\dfrac{dy}{dx} + \dfrac{y}{x} = \sin x.$

3. $\dfrac{dy}{dx}\sin x - y\cos x = 1.$

4. $\dfrac{dy}{dx} - n\dfrac{y}{x} = e^x x^n.$

5. $\dfrac{dy}{dx} = x - y.$

6. $x\dfrac{dy}{dx} + (1+x)\,y = e^x.$

7. Solve the equation $\frac{dx}{dt} = k(s-x)$, [Art. 61 (c)], as a linear differential equation, and determine the constant from the condition $x = 0$ when $t = 0$.

8. The equation

$$M = k\omega + I\frac{d\omega}{dt}$$

gives the moment M required to rotate a disk on a fixed axis when the disk is subjected to fluid frictional resistances. I denotes the moment of inertia of the disk about the axis, and ω is the angular velocity. Derive a relation between ω and t, and determine the constant by the condition that $\omega = \omega_0$ when $t = 0$.

162. Equations reducible to linear form. By an appropriate transformation, an equation that is not linear can in some cases be reduced to the linear form. Thus the equation

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \tag{1}$$

becomes linear by the substitution $v = f(y)$. Another equation easily reducible to linear form is

$$\frac{dy}{dx} + Py = Qy^n. \tag{2}$$

Multiplying both members by $(1-n)\,y^{-n}$, we obtain

$$(1-n)\,y^{-n}\frac{dy}{dx} + (1-n)\,Py^{1-n} = Q(1-n).$$

If now we substitute v for y^{1-n}, and accordingly dv for $(1-n)y^{-n}\,dy$, we have

$$\frac{dv}{dx} + (1-n)\,Pv = Q(1-n), \tag{3}$$

which is a linear equation.

Ex. $\frac{dy}{dx} + \frac{y}{x} = x^2y^4.$

Multiplying by $-3y^{-4}$, we get

$$-3y^{-4}\frac{dy}{dx} - \frac{3}{x}y^{-3} = -3x^2,$$

which by the substitution $v = y^{-3}$ reduces to

$$\frac{dv}{dx} - \frac{3}{x}v = -3x^2.$$

The solution is

$$\frac{v}{x^3} = -3x + c,$$

or

$$\frac{1}{x^3y^3} + 3x = c.$$

EXERCISES.

1. $\dfrac{dy}{dx} + \dfrac{y}{x} = xy^5.$

2. $\dfrac{dy}{dx} + \dfrac{2y}{x} = x^2y^{-3}.$

3. $\dfrac{dy}{dx} + y = xy^3.$

4. $\cos x \dfrac{dy}{dx} - y \sin x + y^2 = 0.$

5. $\dfrac{dy}{dx} + \dfrac{xy}{x^2 - 1} = \dfrac{x}{\sqrt{y}}.$

6. $y \dfrac{dy}{dx} + xy^3 = x.$

DIFFERENTIAL EQUATIONS OF THE *n*TH ORDER AND OF FIRST DEGREE.

163. Linear differential equations. The linear equation of the nth order has the general form

$$D^ny + P_1 D^{n-1}y + P_2 D^{n-2}y + \ldots + P_{n-1} Dy + P_ny = X, \quad (1)$$

in which the coefficients P_1, P_2, etc., are either constants or functions of x, and the second member X is a constant or a function of x. The symbols D^ny, $D^{n-1}y$, etc., stand for $\dfrac{d^ny}{dx^n}$, $\dfrac{d^{n-1}y}{dx^{n-1}}$, etc. Linear equations with variable coefficients lie beyond the scope of this chapter. We shall deal only with those equations in which P_1, P_2, . . . P_n are constants and in which the second member is zero. For a discussion of the general case where the second member is a function of x, the student is referred to works on differential equations.

Let $y = ce^{mx}$, c being an arbitrary constant; then we have

$$Dy = mce^{mx},\ D^2y = m^2ce^{mx},\ \ldots\ D^ny = m^nce^{mx}.$$

If we substitute these expressions for y, Dy, . . . D^ny in the first member of the equation

$$D^ny + P_1 D^{n-1}y + P_2 D^{n-2}y + \ldots + P_{n-1} Dy + P_ny = 0, \quad (2)$$

the resulting expression is

$$ce^{mx}\,(m^n + P_1m^{n-1} + P_2m^{n-2} + \ldots + P_{n-1}m + P_n). \quad (3)$$

If m is a root of the equation

$$m^n + P_1m^{n-1} + P_2m^{n-2} + \ldots + P_{n-1}m + P_n = 0, \quad (4)$$

then (3) reduces to zero, and the relation $y = ce^{mx}$ satisfies equation (2). In other words, if m_1, m_2, . . . m_n are the n roots of (4), then $y = c_1e^{m_1x}$, $y = c_2e^{m_2x}$, . . . $y = c_ne^{m_nx}$ are particular solutions of (2).

Equation (4), whose roots give these particular solutions, is called the *auxiliary equation* of the given differential equation.

The general solution is obtained by a combination of particular solutions. The following cases may arise:

(a) *When the roots of the auxiliary equation are real and distinct.* If these roots are m_1, m_2, . . . m_n, then $y = c_1e^{m_1x}, y = c_2e^{m_2x}, \ldots$ $y = c_ne^{m_nx}$ are particular solutions, and

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx} \tag{5}$$

is the general solution. For, upon substituting this relation in (2), we get

$$\begin{aligned} &c_1e^{m_1x}[m_1^n + P_1m_1^{n-1} + P_2m_1^{n-2} + \ldots + P_{n-1}m_1 + P_n] \\ &+ c_2e^{m_2x}[m_2^n + P_1m_2^{n-1} + P_2m_2^{n-2} + \ldots + P_{n-1}m_2 + P_n] \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ c_ne^{m_nx}[m_n^n + P_1m_n^{n-1} + P_2m_n^{n-2} + \ldots + P_{n-1}m_n + P_n] = 0. \end{aligned}$$

Each of the expressions in the brackets is zero, since m_1, m_2, etc., are roots of (4); hence (5) is a solution of (2). Moreover, it is a general solution, since it contains the required number of arbitrary constants.

Ex. 1. Solve $D^2y - Dy - 12y = 0$.

The auxiliary equation is

$$m^2 - m - 12 = 0,$$

or

$$(m - 4)(m + 3) = 0,$$

the roots of which are 4 and -3; hence the general solution of the equation is

$$y = c_1e^{4x} + c_2e^{-3x}$$

(b) *When the auxiliary equation has imaginary roots.* If the auxiliary equation has a pair of imaginary roots, as $a + bi$, $a - bi$ (where $i = \sqrt{-1}$), the solution may be expressed in real terms as follows: The part of the solution corresponding to these two roots is

$$y = c_1e^{(a+bi)x} + c_2e^{(a-bi)x} = e^{ax}(c_1e^{ibx} + c_2e^{-ibx}). \tag{6}$$

But we have

$$\begin{aligned} e^{ibx} &= \cos bx + i\sin bx, \\ e^{-ibx} &= \cos bx - i\sin dx. \end{aligned}$$

(See Ex. 9, p. 346.)

Substituting these values in (6), we get

$$\begin{aligned} y &= e^{ax}[c_1(\cos bx + i\sin bx) + c_2(\cos bx - i\sin bx)] \\ &= e^{ax}[(c_1 + c_2)\cos bx + i(c_1 - c_2)\sin bx] \\ &= e^{ax}[A\cos bx + B\sin bx], \end{aligned} \tag{7}$$

where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$.

Ex. 2. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0.$

The roots of the auxiliary equation

$$m^2 - 4m + 5$$

are $2 + i$ and $2 - i$.

The solution is therefore $y = e^{2x}(A\cos x + B\sin x)$.

(c) *When the auxiliary equation has multiple roots.* Suppose two of the roots of the auxiliary equation to be equal. For example, let $m_1 = m_2$. In this case the sum $c_1e^{m_1x} + c_2e^{m_2x}$ obtained by the usual process reduces to $e^{m_1x}(c_1 + c_2) = c'e^{m_1x}$, and the two constants are really equivalent to a single constant. Hence the solution having only $n - 1$ arbitrary constants is not the general solution. We may arrive at the general solution by the following device:

Let $m_2 = m_1 + h$, where h is some number that may be made to approach zero. We have as the solution, therefore,

$$\begin{aligned} y &= c_1e^{m_1x} + c_2e^{(m_1+h)x} + \dots + c_ne^{m_nx} \\ &= e^{m_1x}(c_1 + c_2e^{hx}) + \dots + c_ne^{m_nx}. \end{aligned} \tag{8}$$

We now expand e^{hx} by Maclaurin's development in finite form and substitute the result in (8). The result is

$$\begin{aligned} y &= e^{m_1x}\left[c_1 + c_2\left(1 + hx + \frac{1}{2!}h^2x^2 + \dots + \frac{e^{\theta hx}}{n!}h^nx^n\right)\right] \\ &\qquad + \dots + c_ne^{m_nx} \\ &= e^{m_1x}\left[c_1 + c_2 + c_2hx\left(1 + \tfrac{1}{2}hx + \dots + \frac{e^{\theta hx}}{n!}h^{n-1}x^{n-1}\right)\right] \\ &\qquad + \dots + c_ne^{m_nx} \\ &= e^{m_1x}\left[A + Bx\left(1 + \tfrac{1}{2}hx + \dots + \frac{e^{\theta hx}}{n!}h^{n-1}x^{n-1}\right)\right] \\ &\qquad + \dots + c_ne^{m_nx}, \end{aligned} \tag{9}$$

where $A = c_1 + c_2$, and $B = c_2h$.

By making h approach zero, we make the root m_2 approach the root m_1; hence the limit of the second member of (9) as h approaches zero gives the required solution. That is,

$$y = \underset{h \doteq 0}{L}\, e^{m_1 x}\left[A + Bx\left(1 + \tfrac{1}{2}hx + \ldots + \frac{e^{\theta hx}}{n!}h^{n-1}x^{n-1}\right)\right] + \ldots + c_n e^{m_n x}$$

$$= e^{m_1 x}\left[A + Bx\, \underset{h \doteq 0}{L}\left(1 + \tfrac{1}{2}hx + \ldots + \frac{e^{\theta hx}}{n!}h^{n-1}x^{n-1}\right)\right]$$

$$+ \ldots + c_n e^{m_n x}. \tag{10}$$

But, we have

$$\underset{h \doteq 0}{L}\left(1 + \tfrac{1}{2}hx + \ldots + \frac{e^{\theta hx}}{n!}h^{n-1}x^{n-1}\right) = 1.$$

Hence (10) becomes in the limit

$$y = e^{m_1 x}(A + Bx) + \ldots + c_n e^{m_n x}. \tag{11}$$

If k of the roots are equal, a similar process leads to the solution

$$y = e^{m_1 x}(A + Bx + Cx^2 + \ldots + Kx^k) + \ldots + c_n e^{m_n x}.$$

Ex. 3. $D^3y - D^2y - 5\,Dy - 3y = 0.$

The roots of the auxiliary equation are

$$-1,\ -1 \text{ and } 3.$$

Hence the solution is

$$y = e^{-x}(c_1 + c_2 x) + c_3 e^{3x}.$$

If some of the imaginary roots are equal, for example, if $m_1 = m_2 = a + ib$, and $m_3 = m_4 = a - ib$, the part of the solution corresponding to these roots is

$$y = (c_1 + c_2 x)\, e^{(a+ib)x} + (c_3 + c_4 x)\, e^{(a-ib)x}$$

and this reduces to

$$y = e^{ax}\left[(A_1 + B_1 x)\cos bx + (A_2 + B_2 x)\sin Bx\right].$$

Ex. 4. $D^4y - 12\,D^3y + 26\,D^2y - 40\,Dy + 25\,y.$

The roots of the auxiliary equation are $2 + i,\ 2 + i,\ 2 - i,\ 2 - i,$

whence $$y = e^{2x}\left[(c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x\right].$$

EXERCISES.

1. $D^2y + 5\,Dy + 6\,y = 0.$
2. $D^2y + 2\,Dy - 15\,y = 0.$
3. $D^3y - 5\,D^2y + 3\,Dy + 9\,y = 0.$
4. $D^4y + 4\,D^3y + 6\,D^2y + 4\,Dy + y = 0.$
5. $D^3y - y = 0.$
6. $D^2y + 4\,y = 0.$

7. $D^4y - 4D^3y + 10D^2y - 12Dy + 5y = 0.$

8. $D^4y - 4D^3y + 14D^2y - 20Dy + 25y = 0.$

9. $D^4y + 4aD^2y + 4a^2y = 0.$

10. $D^3y - Dy = 0.$

11. $D^3y - 2D^2y - 8Dy = 0.$

12. $4D^3y - 3Dy + y = 0.$

164. Special equations of higher order. In the present article we shall discuss certain differential equations of higher order, which are of interest because of the frequency with which they occur in applications of mathematics.

1. Equations of the type

$$\frac{d^ny}{dx^n} = f(x) \tag{1}$$

are solved by successive integration. See Art. 66. Each integration introduces an arbitrary constant.

2. Equations in the form

$$\frac{d^ny}{dx^n} = f(y) \tag{2}$$

can in general be integrated only for $n = 1$ and $n = 2$.

To integrate

$$\frac{dy}{dx} = f(y), \tag{3}$$

we separate the variables. Thus,

$$\frac{dy}{f(y)} = dx,$$

whence

$$\int \frac{dy}{f(y)} = x + c.$$

To integrate

$$\frac{d^2y}{dx^2} = f(y), \tag{4}$$

we multiply both sides of the equation by $2\frac{dy}{dx}$ We have then

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx}$$

which may be written

$$2\frac{dy}{dx}\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2f(y)\frac{dy}{dx},$$

or

$$2\frac{dy}{dx}\,d\left(\frac{dy}{dx}\right) = 2f(y)\,dy.$$

Integrating we obtain

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\,dy + C,$$

which can be reduced to the same form as equation (3).

3. Equations of the form

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots \frac{dy}{dx}\right) = 0,$$

which involve only derivatives, may be reduced to equations of the next lower order by the substitution $p = \frac{dy}{dx}$. The same device may be used whenever the dependent variable is absent from the equation.

Ex. Solve $\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$ This equation has an application in the theory of potential.

Let $p = \frac{dV}{dr}$; then $\frac{d^2V}{dr^2} = \frac{dp}{dr}$, and the equation upon the substitution of these values reduces to the form

$$\frac{dp}{dr} + \frac{2}{r}p = 0.$$

Separating the variables, we get

$$\frac{dp}{p} + \frac{2\,dr}{r} = 0,$$

whence $$\log p + 2\log r = \log c,$$

or $$p = \frac{dV}{dr} = \frac{c}{r^2}.$$

Integrating a second time, we obtain

$$V = -\frac{c}{r} + c'.$$

4. Equations that do not contain x directly may be reduced to lower order by the substitutions $\frac{dy}{dx} = p$, $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$, and so on.

EXERCISES.

1. $\dfrac{d^3y}{dx^3} = x + 3.$

2. $\dfrac{d^2y}{dx^2} = e^x + \sin x.$

3. $\dfrac{d^2y}{dx^2} = e^{2y}.$

4. $\dfrac{d^2y}{dx^2} = ay + b.$

5. $\dfrac{d^2y}{dx^2} - 3\left(\dfrac{dy}{dx}\right)^2 = 0.$

6. $(1 - y)\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = 0.$

7. $(1 + x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0.$

8. $\dfrac{d^2V}{dr^2} + \dfrac{1}{r}\dfrac{dV}{dr} = 0.$

APPLICATIONS OF DIFFERENTIAL EQUATIONS.

165. Rectilinear motion. The fundamental law governing the rectilinear motion of a body is expressed by the equation

$$F = ma = m\frac{d^2s}{dt^2}, \tag{1}$$

in which m denotes the mass of the body, a the acceleration of its centroid, and F the force acting along the line of motion. The following are a few of the applications of this law:

(a) *Motion in a resisting medium.* When a body falls vertically in a resisting medium, as air, the resistance of the air is a function of the velocity. It is usual to assume it proportional to the square of the velocity. For a free fall in vacuo the equation of motion is

$$F = m\frac{d^2s}{dt^2} = mg;$$

hence for resisted motion,

$$F = m\frac{d^2s}{dt^2} = mg - kv^2. \tag{1}$$

Since $\dfrac{d^2s}{dt^2} = \dfrac{dv}{dt}$, we may write (1) in the form

$$\frac{dv}{dt} + \frac{k}{m}v^2 = g, \tag{2}$$

an equation of the first order with variables separable.

If we place $\dfrac{mg}{k} = V^2$, (2) reduces to

$$\frac{dv}{V^2 - v^2} = \frac{g}{V^2}\,dt,$$

whence

$$\frac{1}{2V}\log\frac{V + v}{V - v} = \frac{gt}{V^2} + c.$$

If the body falls from rest, $v = 0$ when $t = 0$, whence $c = 0$. We have then

$$\frac{V + v}{V - v} = e^{\frac{2gt}{V}}.$$

or

$$v = V \frac{e^{\frac{2gt}{V}} - e^{-\frac{2gt}{V}}}{e^{\frac{2gt}{V}} + e^{-\frac{2gt}{V}}} = V \tanh \frac{2\,gt}{V}. \tag{3}$$

From (3) it appears that V is the limiting value that the velocity v approaches as the time increases indefinitely; hence V is called the *terminal velocity* of the body.

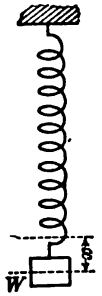

Fig. 147.

(b) *Harmonic vibrations.* Suppose a body of weight W, Fig. 147, suspended by means of a spring or elastic cord. If the body is depressed below the position of equilibrium and then released, what will be the subsequent motion? Let P be the force required to stretch the spring one unit in length, and let a be the extension of the spring due to the weight W; then $W = Pa$. When the body is at a distance x below its position of equilibrium, the upward pull of the spring is Px; hence taking the positive direction of x downwards, the equation of motion is

$$m \frac{d^2x}{dt^2} = W - Px. \tag{1}$$

By the substitutions

$$W = mg, \quad P = \frac{W}{a} = \frac{mg}{a},$$

this equation takes the form

$$\frac{d^2x}{dt^2} + \frac{g}{a}(x - a) = 0, \tag{2}$$

the solution of which is

$$x - a = c_1 \cos \sqrt{\frac{g}{a}}\,t + c_2 \sin \sqrt{\frac{g}{a}}\,t. \tag{3}$$

Let b be the displacement of the body below the equilibrium position at the beginning of motion. Then for $x = b$, $t = 0$, and $\frac{dx}{dt} = 0$. From these conditions we find

$$c_1 = b - a, \qquad c_2 = 0,$$

and substituting these values of c_1 and c_2 in (3), we obtain finally

$$x - a = (b - a) \cos \sqrt{\frac{g}{a}}\, t. \tag{4}$$

This equation shows that the body will oscillate through the distance $b - a$ on both sides of the position of equilibrium, and that the motion is harmonic.

If the spring is horizontal and the body rests on a smooth, frictionless plane, the weight W does not enter into consideration. and the equation of motion becomes simply

$$m \frac{d^2x}{dt^2} = -Px,$$

or

$$\frac{d^2x}{dt^2} + \frac{P}{m} x = 0. \tag{5}$$

The solution of this equation is

$$x = c_1 \cos \sqrt{\frac{P}{m}}\, t + c_2 \sin \sqrt{\frac{P}{m}}\, t. \tag{6}$$

If $x = b$ when $t = 0$ and $\frac{dx}{dt} = 0$, we readily find

$$c_1 = b, \quad c_2 = 0.$$

Hence (6) becomes

$$x = b \cos \sqrt{\frac{P}{m}}\, t. \tag{7}$$

This equation shows that the body will oscillate harmonically through a distance 2 b.

(c) *Damped vibration.* If the motion of the vibrating body is retarded by the friction of the fluid medium surrounding the body, the frictional resistance of the medium must be added to the force Px due to the spring tension. Experiment shows that this force is proportional to the velocity; hence the equation of motion becomes (for the spring horizontal)

$$m \frac{d^2x}{dt^2} = -Px - k \frac{dx}{dt}.$$

This equation takes the form

$$\frac{d^2x}{dt^2} + 2A \frac{dx}{dt} + B^2 x = 0, \tag{1}$$

where $$2A = \frac{k}{m}, \text{ and } B^2 = \frac{P}{m}.$$

The roots of the auxiliary equation

$$m^2 + 2Am + B^2 = 0$$

are $$-A \pm \sqrt{A^2 - B^2}.$$

There are four cases to be considered, depending upon the value of A, that is, upon the magnitude of the frictional resistance.

1. If $A = 0$, the roots of the auxiliary equation are $\pm iB$, and the solution is that previously obtained in (*b*).

2. If $0 < A < B$, the roots are of the form $-\alpha \pm i\beta$. In this case the solution is

$$x = e^{-\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t). \tag{2}$$

3. If $A = B$, the roots are equal, and the solution is

$$x = (c_1 + c_2 t)\, e^{-At}. \tag{3}$$

4. If $A > B$, the roots are real and unequal, and may be denoted by $-\alpha$ and $-\beta$. The solution is in this case

$$x = c_1 e^{-\alpha t} + c_2 e^{-\beta t}. \tag{4}$$

Fig. 148, from Perry's *Calculus for Engineers*, shows the displacement time curves for each of these cases. Curve 1 corresponds to $A = 0$, that is, no friction. Curve 2 corresponds to the case $A < B$; here the vibrations are apparent, but the amplitude is growing smaller. As the frictional resistance is still further increased, curves 3 and 4 are obtained.

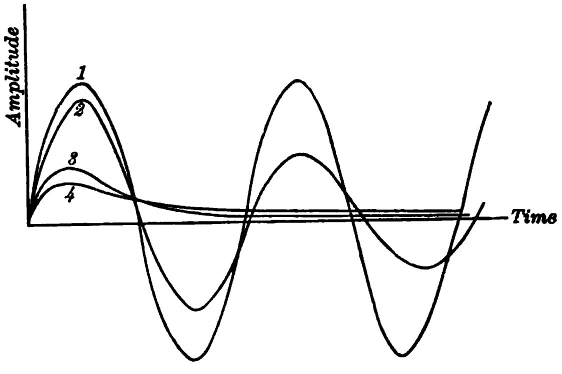

Fig. 148.

This damping effect is practically always present. Sometimes the resistance is purposely made large by causing the body to move in a liquid, as oil or water.

Ex. Take the values $m = 1$, $P = 4$, $k = 0$, 0.4, 4, and 10. Let $x = 5$ when $t = 0$, and $\frac{dx}{dt} = 0$. Taking in succession the four values of k, draw the four curves showing the displacement on a time base.

166. Motion of a simple pendulum. The ideal simple pendulum consists of a particle of mass m constrained to move in a circular arc. For the purpose of concrete representation we may suppose the particle to be a small, heavy ball P, Fig. 149, supported from a fixed point O by fine wire OP.

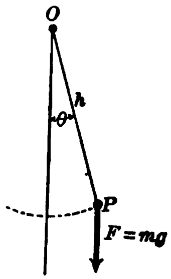

Fig. 149.

The equation of motion for a body constrained to rotate about a fixed axis is

$$I \frac{d^2\theta}{dt^2} = M, \qquad (1)$$

in which I denotes the moment of inertia of the body about the axis, and M the moment of the force producing the rotation relative to the axis. In the case of the pendulum the force that causes the motion is the weight mg of the mass m. The moment of this force about an axis through the point O is

$$-\,mgh \sin\theta,$$

where h denotes the length OP. The moment of inertia of the mass m relative to O is mh^2. Hence from (1) we have

$$mh^2 \frac{d^2\theta}{dt^2} = -\,mgh \sin\theta,$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{h} \sin\theta. \qquad (2)$$

If the arc traversed by the particle is small, θ may be substituted for $\sin\theta$. Therefore we have approximately the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{h}\theta = 0, \qquad (3)$$

the solution of which is

$$\theta = c_1 \cos t\sqrt{\frac{g}{h}} + c_2 \sin t\sqrt{\frac{g}{h}}. \qquad (4)$$

Let $t = 0$ and $\dfrac{d\theta}{dt} = 0$ when $\theta = \beta$. The constants are then found to be $c_1 = \beta, c_2 = 0$, and (4) becomes

$$\theta = \beta \cos t \sqrt{\frac{g}{h}}\,. \tag{5}$$

This equation shows that the motion is harmonic. As an exercise the student may show that the time of passing from one extreme position to the other is $\pi \sqrt{\dfrac{h}{g}}$.

When the resistance of the medium in which the pendulum moves is taken into account, the equation of motion becomes

$$\frac{d^2\theta}{dt^2} + 2\,P\,\frac{d\theta}{dt} + Q\theta = 0,$$

the term $2\,P\dfrac{d\theta}{dt}$ arising from the frictional resistance, which is assumed proportional to the angular velocity. This equation, it will be observed, is of the same form as that for a damped vibration, and has therefore a solution of the same form. This same equation applies to the damped vibrations of a galvanometer needle.

167. Flow of heat. Heat flows steadily through a long bar having a cross section A and perimeter p. If τ is the temperature at a cross section whose distance from the hot end of the bar is x, the equation of flow is

$$KA\,\frac{d^2\tau}{dx^2} = Ep\tau, \tag{1}$$

in which K denotes the conductivity and E the surface emissivity of the bar.* Denoting by m^2 the factor $\dfrac{Ep}{KA}$, (1) takes the form

$$\frac{d^2\tau}{dx^2} = m^2\tau, \tag{2}$$

whence

$$\tau = c_1e^{mx} + c_2e^{-mx}. \tag{3}$$

This equation gives the law of variation of temperature along the bar. The constants may be obtained from initial conditions.

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* See Preston's *Theory of Heat*, Art. 255.

MISCELLANEOUS EXERCISES.

1. Find the differential equation of all circles of fixed radius a which have their centers on the Y-axis.

2. Find the curves which have the property that the length of the normal varies as the square of the ordinate.

3. Solve the following differential equations:

$$\text{(a)}\quad (2\sqrt{xy} - x)\,dy + y\,dx = 0.$$

$$\text{(b)}\quad e^y \sin x \frac{dy}{dx} = e^x - e^y \cos x.$$

$$\text{(c)}\quad \frac{dy}{dx} + \frac{1 - 2x}{x^2}\,y = 1.$$

4. Find the solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2 - \frac{m^2}{x^2} = 0.$$

Suggestion: Solve for $\frac{dy}{dx}$ and integrate.

5. Show that the equation

$$y = px + f(p),$$

where $p = \frac{dy}{dx}$, has the general solution

$$y = cx + f(c).$$

6. From the simultaneous equations

$$\frac{dx}{dt} + 3x - 2y = 0,$$

$$\frac{dy}{dt} - x + 2y = 0,$$

find a solution expressing x and y respectively as functions of t.

Suggestion: Differentiate one of the equations with respect to t, and eliminate either x or y.

7. Show that the homogeneous linear equation

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + P_{n-1}\, x \frac{dy}{dx} + P_n y = 0$$

can be reduced to a linear equation with constant coefficients by the substitution $x = e^z$. In this way solve the equations

$$\text{(a)}\quad x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

$$\text{(b)}\quad x^3 \frac{d^3y}{dx^3} - 5x^2 \frac{d^2y}{dx^2} + 10y = 0.$$

8. The equation of motion of a particle subjected to an attractive force varying inversely as the square of the distance is $\frac{d^2x}{dt^2} = -\frac{k}{x^2}$. (a) If the initial distance from the center of force is a, find the time required for the particle to move to the center, assuming that it starts from rest. (b) Find the velocity at any point of the path.

9. The equation of motion of a particle in a smooth, straight tube which revolves with constant angular speed ω in a horizontal plane is

$$\frac{d^2\rho}{dt^2} - \omega^2\rho = 0.$$

Find the solution of this equation, and determine the constants from the condition that $\rho = a$, and $\frac{d\rho}{dt} = V_0$, when $t = 0$.

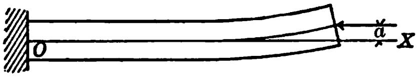

Fig. 150.

10. A strut, Fig. 150, is subjected to a force P acting parallel to the axis OX. The deflection of the end is a. From the theory of bending, the curve assumed by the axis must be such as to satisfy the differential equation

$$\frac{d^2y}{dx^2} = \frac{P}{EI}(a - y),$$

in which E denotes the coefficient of elasticity, and I the second moment of the cross section. (a) Show that the equation of the axis is

$$y = a\left(1 - \cos x\sqrt{\frac{P}{EI}}\right).$$

(b) If the length of the strut is l, show that

$$P = \frac{\pi^2}{4}\frac{EI}{l^2}.$$

This formula is *Euler's formula* for long columns. It gives the load P at which incipient bending occurs.

INTEGRALS FOR REFERENCE

GENERAL FORMULAS

a. $\int au\, du = a\int u\, du.$

b. $\int (u_1 + u_2 + \ldots + u_n)\, du = \int u_1\, du + \int u_2\, du + \ldots + \int u_n\, du.$

c. $\int u\, dv = uv - \int v\, du.$

I. FUNDAMENTAL FORMS

For convenience of reference, the forms given on pp. 124, 125, are here repeated. Extensions of these forms are given later in their proper sections.

1. $\int x^n\, dx = \dfrac{x^{n+1}}{n+1}.$

2. $\int \cos x\, dx = \sin x.$

3. $\int \sin x\, dx = -\cos x.$

4. $\int \sec^2 x\, dx = \tan x.$

5. $\int \csc^2 x\, dx = -\cot x.$

6. $\int \sec x \tan x\, dx = \sec x.$

7. $\int \csc x \cot x\, dx = -\csc x.$

8. $\int \dfrac{dx}{\sqrt{1 - x^2}} = \text{arc sin } x.$

9. $\int \dfrac{dx}{1 + x^2} = \text{arc tan } x.$

10. $\int \dfrac{dx}{x\sqrt{x^2 - 1}} = \text{arc sec } x.$

11. $\int \dfrac{dx}{\sqrt{2x - x^2}} = \text{arc vers } x.$

12. $\int a^x\, dx = \dfrac{a^x}{\log a}.$

13. $\int e^x\, dx = e^x.$

14. $\int \dfrac{dx}{x} = \log x.$

15. $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a} \log \dfrac{x - a}{x + a}.$

16. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right).$

17. $\int \tan x\, dx = \log \sec x.$

18. $\int \cot x\, dx = \log \sin x.$

19. $\int \csc x\, dx = \log \tan \dfrac{x}{2}.$

20. $\int \sec x\, dx = \log \tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$

II. RATIONAL ALGEBRAIC FUNCTIONS

1. Expressions containing $(a + bx)$ and $(a + bx^n)$

21. $\int \frac{x\,dx}{a + bx} = \frac{1}{b^2}[a + bx - a \log (a + bx)].$

22. $\int \frac{x\,dx}{(a + bx)^2} = \frac{1}{b^2}\left[\log (a + bx) + \frac{a}{a + bx}\right].$

23. $\int \frac{x^2\,dx}{a + bx} = \frac{1}{b^3}[\tfrac{1}{2} (a + bx)^2 - 2a(a + bx) + a^2 \log (a + bx)].$

24. $\int \frac{x^2\,dx}{(a + bx)^2} = \frac{1}{b^3}\left[a + bx - 2a \log (a + bx) - \frac{a^2}{a + bx}\right].$

25. $\int \frac{dx}{x(a + bx)} = -\frac{1}{a} \log \frac{a + bx}{x}.$

26. $\int \frac{dx}{x^2(a + bx)} = -\frac{1}{ax} + \frac{b}{a^2} \log \frac{a + bx}{x}.$

27. $\int \frac{dx}{x(a + bx)^2} = \frac{1}{a(a + bx)} - \frac{1}{a^2} \log \frac{a + bx}{x}.$

28. $\int \frac{dx}{x(a + bx^n)} = \frac{1}{an} \log \frac{x^n}{a + bx^n}.$

29. $\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \arctan x\sqrt{\frac{b}{a}}.$

30. $\int \frac{x\,dx}{a + bx^2} = \frac{1}{2b} \log \frac{a + bx^2}{b}.$

31. $\int \frac{x^2\,dx}{a + bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a + bx^2}.$

32. $\int \frac{dx}{(a + bx^2)^2} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a}\int \frac{dx}{a + bx^2}.$

33. $\int \frac{dx}{(a+bx^2)^n} = \frac{1}{2a(n-1)}\left[\frac{x}{(a+bx^2)^{n-1}} + (2n-3)\int \frac{dx}{(a + bx^2)^{n-1}}\right]$

2. Expressions containing $a + bx + cx^2$

Let $X = a + bx + cx^2$.

34. $\int \frac{dx}{X} = \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2cx + b}{\sqrt{4ac - b^2}}$, when $4ac > b^2$.

35. $\int \frac{dx}{X} = \frac{1}{\sqrt{b^2 - 4ac}} \log \frac{b + 2cx - \sqrt{b^2 - 4ac}}{b + 2cx + \sqrt{b^2 - 4ac}}$, when $b^2 > 4ac$.

36. $\int \frac{dx}{X^2} = \frac{2cx + b}{X(4ac - b^2)} + \frac{2c}{4ac - b^2}\int \frac{dx}{X}.$

37. $\int \frac{dx}{X^3} = \frac{2cx + b}{X(4ac - b^2)}\left(\frac{1}{2X} + \frac{3c}{4ac - b^2}\right) + \frac{6c^2}{(4ac - b^2)^2}\int \frac{dx}{X}.$

38. $\int \frac{x\,dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}.$

39. $\int \frac{x\,dx}{X^2} = - \frac{2a + bx}{X(4ac - b^2)} - \frac{b}{4ac - b^2} \int \frac{dx}{X}.$

40. $\int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}.$

41. $\int \frac{dx}{x^2X} = \frac{b}{2a^2} \log \frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a}\right) \int \frac{dx}{X}.$

III. IRRATIONAL ALGEBRAIC FUNCTIONS

1. Expressions containing $\sqrt{a + bx}$

42. $\int \sqrt{a + bx}\,dx = \frac{2}{3b}(a + bx)^{\frac{3}{2}}.$

43. $\int x\sqrt{a + bx}\,dx = - \frac{2}{15b^2}(2a - 3bx)(a + bx)^{\frac{3}{2}}.$

44. $\int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b}\sqrt{a + bx}.$

45. $\int \frac{x\,dx}{\sqrt{a + bx}} = - \frac{2}{3b^2}(2a - bx)\sqrt{a + bx}.$

46. $\int \frac{dx}{x\sqrt{a + bx}} = \frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}\right).$

47. $\int \frac{dx}{x\sqrt{bx - a}} = \frac{2}{\sqrt{a}} \arctan \sqrt{\frac{bx - a}{a}}.$

48. $\int \frac{\sqrt{a + bx}\,dx}{x} = 2\sqrt{a + bx} + a \int \frac{dx}{x\sqrt{a + bx}}.$

2. Expressions containing $\sqrt{x^2 \pm a^2}$

49. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \log (x + \sqrt{x^2 \pm a^2}).$

50. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log (x + \sqrt{x^2 \pm a^2}).$

51. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arc\,sec} \frac{x}{a}.$

52. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \log \frac{x}{a + \sqrt{x^2 + a^2}}.$

53. $\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}(x^2 \pm a^2)^{\frac{3}{2}}.$

54. $$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}.$$

55. $$\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - \operatorname{arc\,sec}\frac{x}{a}.$$

56. $$\int (x^2 \pm a^2)^{\frac{3}{2}}\,dx = \frac{x}{8}(2x^2 \pm 5a^2)\sqrt{x^2 \pm a^2} + \frac{3a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right).$$

57. $$\int \frac{dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \pm\frac{x}{a^2\sqrt{x^2 \pm a^2}}.$$

58. $$\int \frac{x\,dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = -\frac{1}{\sqrt{x^2 \pm a^2}}.$$

59. $$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{8}(2x^2 \pm a^2)\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right).$$

60. $$\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right).$$

61. $$\int \frac{x^2\,dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2 \pm a^2}} + \log\left(x + \sqrt{x^2 \pm a^2}\right).$$

62. $$\int \frac{\sqrt{x^2 \pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log\left(x + \sqrt{x^2 \pm a^2}\right).$$

63. $$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2 x}.$$

64. $$\int \frac{dx}{x^3\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2x^2} + \frac{1}{2a^3}\operatorname{arc\,sec}\frac{x}{a}.$$

65. $$\int \frac{dx}{x^3\sqrt{x^2 + a^2}} = -\frac{\sqrt{x^2 + a^2}}{2a^2x^2} + \frac{1}{2a^3}\log\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right).$$

66. $$\int (x^2 \pm a^2)^{\frac{n}{2}}\,dx = \frac{x(x^2 \pm a^2)^{\frac{n}{2}}}{n+1} \pm \frac{na^2}{n+1}\int (x^2 \pm a^2)^{\frac{n}{2}-1}dx.$$

3. Expressions containing $\sqrt{a^2 - x^2}$

67. $$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\operatorname{arc\,sin}\frac{x}{a}.$$

68. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \operatorname{arc\,sin}\frac{x}{a}.$$

69. $$\int (a^2 - x^2)^{\frac{3}{2}}\,dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8}\operatorname{arc\,sin}\frac{x}{a}.$$

70. $$\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2 - x^2}}.$$

71. $$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\log\frac{a + \sqrt{a^2 - x^2}}{x}.$$

72. $$\int\frac{dx}{x^2\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{a^2x}.$$

73. $$\int\frac{dx}{x^3\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{2\,a^2x^2}-\frac{1}{2\,a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}.$$

74. $$\int x\sqrt{a^2-x^2}\,dx=-\tfrac{1}{3}\sqrt{(a^2-x^2)^3}.$$

75. $$\int x^2\sqrt{a^2-x^2}\,dx=\frac{x}{8}(2\,x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\text{ arc sin }\frac{x}{a}.$$

76. $$\int\frac{x\,dx}{\sqrt{a^2-x^2}}=-\sqrt{a^2-x^2}.$$

77. $$\int\frac{x^2dx}{\sqrt{a^2-x^2}}=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\text{ arc sin }\frac{x}{a}.$$

78. $$\int\frac{x\,dx}{(a^2-x^2)^{\frac{3}{2}}}=\frac{1}{\sqrt{a^2-x^2}}.$$

79. $$\int\frac{x^2\,dx}{(a^2-x^2)^{\frac{3}{2}}}=\frac{x}{\sqrt{a^2-x^2}}-\text{arc sin }\frac{x}{a}.$$

80. $$\int\frac{\sqrt{a^2-x^2}\,dx}{x}=\sqrt{a^2-x^2}-a\,\log\frac{a+\sqrt{a^2-x^2}}{x}$$

81. $$\int\frac{\sqrt{a^2-x^2}}{x^2}dx=-\frac{\sqrt{a^2-x^2}}{x}-\text{arc sin }\frac{x}{a}.$$

82. $$\int(a^2-x^2)^{n}\,dx=\frac{x(a^2-x^2)^{\frac{n}{2}}}{n+1}+\frac{a^2n}{n+1}\int(a^2-x^2)^{\frac{n}{2}-1}\,dx.$$

83. $$\int\frac{x^m\,dx}{\sqrt{a^2-x^2}}=-\frac{x^{m-1}\sqrt{a^2-x^2}}{m}+\frac{(m-1)\,a^2}{m}\int\frac{x^{m-2}\,dx}{\sqrt{a^2-x^2}}.$$

84. $$\int x^m\sqrt{a^2-x^2}\,dx=\frac{x^{m+1}\sqrt{a^2-x^2}}{m+2}+\frac{a^2}{m+2}\int\frac{x^m\,dx}{\sqrt{a^2-x^2}}.$$

85. $$\int\frac{dx}{x^m\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{(m-1)\,a^2x^{m-1}}+\frac{m-2}{a^2(m-1)}\int\frac{dx}{x^{m-2}\sqrt{a^2-x^2}}.$$

86. $$\int\frac{\sqrt{a^2-x^2}}{x^m}dx=-\frac{\sqrt{a^2-x^2}}{(m-2)\,x^{m-1}}-\frac{a^2}{m-2}\int\frac{dx}{x^m\sqrt{a^2-x^2}}.$$

4. Expressions containing $\sqrt{a+bx+cx^2}$

The integrals contained in the preceding sections are for the most part special cases of the following integrals.

Let $X=a+bx+cx^2$. Then

87. $$\int\frac{dx}{\sqrt{X}}=\frac{1}{\sqrt{c}}\log\left(\sqrt{X}+x\sqrt{c}+\frac{b}{2\sqrt{c}}\right),\text{ if } c>0.$$

88. $$\int\frac{dx}{\sqrt{X}}=\frac{1}{\sqrt{-c}}\text{ arc sin }\left(\frac{-b-2cx}{\sqrt{b^2-4ac}}\right),\text{ if } c<0.$$

89. $\int \sqrt{X}\,dx = \frac{b + 2\,cx}{4\,c}\sqrt{X} - \frac{b^2 - 4\,ac}{8\,c}\int \frac{dx}{\sqrt{X}}.$

90. $\int X\sqrt{X}\,dx = \frac{b + 2\,cx}{8\,c}\sqrt{X}\left[X - \frac{3\,(b^2 - 4\,ac)}{8\,c}\right] + \frac{3\,(b^2 - 4\,ac)^2}{128\,c^2}\int \frac{dx}{\sqrt{X}}.$

91. $\int \frac{dx}{X\sqrt{X}} = -\frac{2\,(b + 2\,cx)}{(b^2 - 4\,ac)\sqrt{X}}.$

92. $\int \frac{dx}{X^2\sqrt{X}} = -\frac{2\,(b + 2\,cx)}{3\,(b^2 - 4\,ac)\sqrt{X}}\left[\frac{1}{X} - \frac{8\,c}{b^2 - 4\,ac}\right].$

93. $\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2\,c}\int \frac{dx}{\sqrt{X}}.$

94. $\int \frac{x\,dx}{X\sqrt{X}} = \frac{2\,(bx + 2\,a)}{(b^2 - 4\,ac)\sqrt{X}}.$

95. $\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2\,c} - \frac{3\,b}{4\,c}\right)\sqrt{X} + \frac{3\,b^2 - 4\,ac}{8\,c^2}\int \frac{dx}{\sqrt{X}}.$

96. $\int x\sqrt{X}\,dx = \frac{X^{\frac{3}{2}}}{3\,c} - \frac{b}{2\,c}\int \sqrt{X}\,dx.$

97. $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a} + \sqrt{X}}{x} + \frac{b}{2\sqrt{a}}\right)$, if $a > 0$.

98. $\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}}\arcsin\frac{bx + 2\,a}{x\sqrt{b^2 - 4\,ac}}$, if $a < 0$.

99. $\int \frac{dx}{x\sqrt{X}} = -\frac{2}{bx}\sqrt{X}$, if $a = 0$.

100. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2\,a}\int \frac{dx}{x\sqrt{X}}.$

101. $\int \frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + a\int \frac{dx}{x\sqrt{X}}.$

102. $\int \frac{\sqrt{X}\,dx}{x^2} = -\frac{1}{x}\sqrt{X} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + c\int \frac{dx}{\sqrt{X}}.$

IV. TRANSCENDENTAL FUNCTIONS

103. $\int \sin^2 x\,dx = \frac{1}{2}\,x - \frac{1}{2}\sin x\cos x.$

104. $\int \sin^3 x\,dx = -\frac{1}{3}\cos x\,(\sin^2 x + 2).$

105. $\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n - 1}{n}\int \sin^{n-2} x\,dx.$

106. $\int \cos^2 x\,dx = \frac{1}{2}\,x + \frac{1}{2}\sin x\cos x.$

107. $\int \cos^3 x\,dx = \frac{1}{3}\sin x\,(\cos^2 x + 2).$

72. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}.$

73. $\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}.$

74. $\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}\sqrt{(a^2-x^2)^3}.$

75. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\text{ arc sin }\frac{x}{a}.$

76. $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}.$

77. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\text{ arc sin }\frac{x}{a}.$

78. $\int \frac{x\,dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{1}{\sqrt{a^2-x^2}}.$

79. $\int \frac{x^2\,dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{a^2-x^2}} - \text{arc sin }\frac{x}{a}.$

80. $\int \frac{\sqrt{a^2-x^2}\,dx}{x} = \sqrt{a^2-x^2} - a\log\frac{a+\sqrt{a^2-x^2}}{x}$

81. $\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - \text{arc sin }\frac{x}{a}.$

82. $\int (a^2-x^2)^{n}dx = \frac{x(a^2-x^2)^{\frac{n}{2}}}{n+1} + \frac{a^2n}{n+1}\int(a^2-x^2)^{\frac{n}{2}-1}dx.$

83. $\int \frac{x^m\,dx}{\sqrt{a^2-x^2}} = -\frac{x^{m-1}\sqrt{a^2-x^2}}{m} + \frac{(m-1)a^2}{m}\int\frac{x^{m-2}\,dx}{\sqrt{a^2-x^2}}.$

84. $\int x^m\sqrt{a^2-x^2}\,dx = \frac{x^{m+1}\sqrt{a^2-x^2}}{m+2} + \frac{a^2}{m+2}\int\frac{x^m\,dx}{\sqrt{a^2-x^2}}.$

85. $\int \frac{dx}{x^m\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{(m-1)a^2x^{m-1}} + \frac{m-2}{a^2(m-1)}\int\frac{dx}{x^{m-2}\sqrt{a^2-x^2}}.$

86. $\int \frac{\sqrt{a^2-x^2}}{x^m}dx = -\frac{\sqrt{a^2-x^2}}{(m-2)x^{m-1}} - \frac{a^2}{m-2}\int\frac{dx}{x^m\sqrt{a^2-x^2}}.$

4. Expressions containing $\sqrt{a+bx+cx^2}$

The integrals contained in the preceding sections are for the most part special cases of the following integrals.

Let $X = a + bx + cx^2$. Then

87. $\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}}\log\left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right)$, if $c > 0$.

88. $\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-c}}\text{ arc sin }\left(\frac{-b-2cx}{\sqrt{b^2-4ac}}\right)$, if $c < 0$.

89. $\int \sqrt{X}\, dx = \frac{b + 2\,cx}{4\,c} \sqrt{X} - \frac{b^2 - 4\,ac}{8\,c} \int \frac{dx}{\sqrt{X}}.$

90. $\int X \sqrt{X}\, dx = \frac{b + 2\,cx}{8\,c} \sqrt{X} \left[X - \frac{3\,(b^2 - 4\,ac)}{8\,c}\right] + \frac{3\,(b^2 - 4\,ac)^2}{128\,c^2} \int \frac{dx}{\sqrt{X}}.$

91. $\int \frac{dx}{X \sqrt{X}} = - \frac{2\,(b + 2\,cx)}{(b^2 - 4\,ac) \sqrt{X}}.$

92. $\int \frac{dx}{X^2 \sqrt{X}} = - \frac{2\,(b + 2\,cx)}{3\,(b^2 - 4\,ac) \sqrt{X}} \left[\frac{1}{X} - \frac{8\,c}{b^2 - 4\,ac}\right].$

93. $\int \frac{x\, dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2\,c} \int \frac{dx}{\sqrt{X}}.$

94. $\int \frac{x\, dx}{X \sqrt{X}} = \frac{2\,(bx + 2\,a)}{(b^2 - 4\,ac) \sqrt{X}}.$

95. $\int \frac{x^2\, dx}{\sqrt{X}} = \left(\frac{x}{2\,c} - \frac{3\,b}{4\,c}\right) \sqrt{X} + \frac{3\,b^2 - 4\,ac}{8\,c^2} \int \frac{dx}{\sqrt{X}}.$

96. $\int x \sqrt{X}\, dx = \frac{X^{\frac{3}{2}}}{3\,c} - \frac{b}{2\,c} \int \sqrt{X}\, dx.$

97. $\int \frac{dx}{x \sqrt{X}} = - \frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{a} + \sqrt{X}}{x} + \frac{b}{2 \sqrt{a}}\right)$, if $a > 0$.

98. $\int \frac{dx}{x \sqrt{X}} = \frac{1}{\sqrt{-a}} \text{arc sin} \frac{bx + 2\,a}{x \sqrt{b^2 - 4\,ac}}$, if $a < 0$

99. $\int \frac{dx}{x \sqrt{X}} = - \frac{2}{bx} \sqrt{X}$, if $a = 0$.

100. $\int \frac{dx}{x^2 \sqrt{X}} = - \frac{\sqrt{X}}{ax} - \frac{b}{2\,a} \int \frac{dx}{x \sqrt{X}}.$

101. $\int \frac{\sqrt{X}\, dx}{x} = \sqrt{X} + \frac{b}{2} \int \frac{dx}{\sqrt{X}} + a \int \frac{dx}{x \sqrt{X}}.$

102. $\int \frac{\sqrt{X}\, dx}{x^2} = - \frac{1}{x} \sqrt{X} + \frac{b}{2} \int \frac{dx}{x \sqrt{X}} + c \int \frac{dx}{\sqrt{X}}.$

IV. TRANSCENDENTAL FUNCTIONS

103. $\int \sin^2 x\, dx = \frac{1}{2} x - \frac{1}{2} \sin x \cos x.$

104. $\int \sin^3 x\, dx = - \frac{1}{3} \cos x\, (\sin^2 x + 2).$

105. $\int \sin^n x\, dx = - \frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx.$

106. $\int \cos^2 x\, dx = \frac{1}{2} x + \frac{1}{2} \sin x \cos x.$

107. $\int \cos^3 x\, dx = \frac{1}{3} \sin x\, (\cos^2 x + 2).$

108. $\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$

109. $\int \tan x\,dx = -\log\cos x = \log\sec x.$

110. $\int \tan^2 x\,dx = \tan x - x.$

111. $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.$

112. $\int \cot x\,dx = \log\sin x.$

113. $\int \cot^2 x\,dx = -\cot x - x.$

114. $\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx.$

115. $\int \sec x\,dx = \log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$

116. $\int \sec^2 x\,dx = \tan x.$

117. $\int \csc x\,dx = \log\tan\frac{x}{2}.$

118. $\int \csc^2 x\,dx = -\cot x.$

119. $\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x.$

120. $\int \sin^n x \cos x\,dx = \frac{1}{n+1}\sin^{n+1} x.$

121. $\int \sin x \cos^n x\,dx = -\frac{1}{n+1}\cos^{n+1} x.$

122. $\int \sin^2 x \cos^2 x\,dx = \frac{x}{8} - \frac{\sin 4x}{32}.$

123. $\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\,dx.$

124. $\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\,dx.$

125. $\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m+1} x \cos^n x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x.$

126. $\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x\,dx.$

127. $\int \frac{dx}{\sin^m x} = -\frac{1}{m-1}\;\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}.$

128. $\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\;\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}.$

129. $\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}}\ \text{arc tan}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$, if $a^2 > b^2$.

130. $$\int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b+a} + \sqrt{b-a}\ \tan\frac{x}{2}}{\sqrt{b+a} - \sqrt{b-a}\ \tan\frac{x}{2}}, \text{ if } a^2 < b^2.$$

131. $$\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \text{ arc tan } \frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}}, \text{ if } a^2 > b^2.$$

132. $$\int \frac{dx}{a + b\sin x} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{a\tan\frac{x}{2} + b - \sqrt{b^2 - a^2}}{a\tan\frac{x}{2} + b + \sqrt{b - a^2}}, \text{ if } a^2 < b^2.$$

133. $\int x \sin x\,dx = \sin x - x\cos x.$

134. $\int x^2 \sin x\,dx = 2x\sin x - (x^2 - 2)\cos x.$

135. $\int x^m \sin x\,dx = -x^m \cos x + m\int x^{m-1}\cos x\,dx.$

136. $\int x\sin^2 x\,dx = \frac{1}{4}(x^2 - 2x\sin x\cos x + \sin^2 x).$

137. $\int x \cos x\,dx = \cos x + x\sin x.$

138. $\int x^2 \cos x\,dx = 2x\cos x + (x^2 - 2)\sin x.$

139. $\int x^m \cos x\,dx = x^m \sin x - m\int x^{m-1}\sin x\,dx.$

140. $\int x\cos^2 x\,dx = \frac{1}{4}(x^2 + 2x\sin x\cos x + \cos^2 x).$

141. $\int x\sin x\cos x\,dx = \frac{1}{4}(x - 2x\cos^2 x + \sin x\cos x).$

142. $$\int \frac{\sin x\,dx}{x^m} = -\frac{1}{m-1}\frac{\sin x}{x^{m-1}} + \frac{1}{m-1}\int\frac{\cos x\,dx}{x^{m-1}}.$$

143. $$\int \frac{\cos x\,dx}{x^m} = -\frac{1}{m-1}\frac{\cos x}{x^{m-1}} - \frac{1}{m-1}\int\frac{\sin x\,dx}{x^{m-1}}.$$

144. $$\int \frac{\sin x\,dx}{x} = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots$$

145. $$\int \frac{\cos x\,dx}{x} = \log x - \frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} - \frac{x^6}{6\cdot 6!} + \cdots$$

146. $\int \text{arc sin } x\,dx = x \text{ arc sin } x + \sqrt{1 - x^2}.$

147. $\int \text{arc cos } x\,dx = x \text{ arc cos } x - \sqrt{1 - x^2}.$

148. $\int \text{arc tan } x\,dx = x \text{ arc tan } x - \frac{1}{2}\log(1 + x^2).$

149. $\int \text{arc cot } x\,dx = x \text{ arc cot } x + \frac{1}{2}\log(1 + x^2).$

150. $\int e^{ax}\,dx = \frac{e^{ax}}{a}$.

151. $\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$.

152. $\int e^{ax}\sin x\,dx = \frac{e^{ax}(a\sin x - \cos x)}{a^2 + 1}$.

153. $\int e^{ax}\cos x\,dx = \frac{e^{ax}(a\cos x + \sin x)}{a^2 + 1}$.

154. $\int \log x\,dx = x\log x - x$.

155. $\int x^m \log x\,dx = x^{m+1}\left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2}\right]$.

DEFINITE INTEGRALS.

156. $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

$= \frac{1\cdot 3\cdot 5\ldots(n-1)}{2\cdot 4\cdot 6\ldots n}\cdot\frac{\pi}{2}$, if n is an even integer.

$= \frac{2\cdot 4\cdot 6\ldots(n-1)}{1\cdot 3\cdot 5\cdot 7\ldots n}$, if n is an odd integer.

157. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k^2\sin^2 x}}$

$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \ldots\right]$, if $k^2 < 1$

158. $\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2 x}\,dx$

$= \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \ldots\right]$, if $k^2 < 1$.

ANSWERS

Art. 3. Pages 4–5.

1. -26; -14; -110. **3.** 1; $\frac{1}{2}\sqrt{2}$; 0; -1.

6. $3-\sqrt{y^2+4}$; $(3-\sqrt{x})^2+4$; $(y^2+4)^2+4$; $3-\sqrt{3-\sqrt{x}}$.

7. $z=-\frac{1}{x}$. **14.** $\cos\theta$; $\sin\theta$.

15. $\sec\theta$.

12. $\sin(x\pm y)$. **16.** $\tan(x-y)$.

Art. 5. Page 10.

1. $y=-\frac{bx+d}{ax+c}$; $x=-\frac{cy+d}{ay+b}$. **4.** $y=4\log\frac{x}{c}$.

2. $y=\sqrt{1-x^2}\sin\beta-x\cos\beta$. **5.** $x=a\operatorname{arc}\cos\frac{a-y}{a}\pm\sqrt{2ay-y^2}$.

3. $v=\left(\frac{c}{p}\right)^{\frac{1}{m}}$. **6.** $u=m\pm\sqrt{m^2-v^2}$.

7. $T=\frac{-B\pm\sqrt{B^2-4C(A-\log p)}}{2(A-\log p)}$.

8. (a) $\sqrt[3]{y}$. (b) $\pm\sqrt{y^2+a^2}$. (c) $\pm\sqrt{e^y-a^2}$.

9. $\frac{1}{2}(e^x-e^{-x})$. **11.** 1.

10. $\frac{e^u-e^{-u}}{e^u+e^{-u}}$. **12.** $\frac{3+5y}{1-2y}$.

Art. 10. Page 20.

2. $x=2, x=3$. **4.** (a) Discontinuous at $x=0$. (b) Discontinuous at $x=0$.

Art. 13. Page 31.

1. $-\frac{4}{5}$. **3.** 0. **5.** 48. **11.** $-\frac{5}{8}$; 0

2. $\frac{1}{2}$. **4.** 1. **6.** $7a^6$. **12.** 1.

Art. 15. Page 40.

1. (a) $(2x-3)\Delta x+(\Delta x)^2$. (b) 0.31.

2. (a) $3x_1^2\Delta x+3x_1(\Delta x)^2+(\Delta x)^3$. (b) $3x_1^2$.

4. $\Delta s=160\,\Delta t+16(\Delta t)^2$. $L\,\frac{\Delta s}{\Delta t}=160$.

5. $\Delta v=3.615021\,\pi$ cu. in. $\Delta s=1.2025\,\pi$ sq. in.

Art. 16. Page 43.

1. $5x^4$.
2. $2x - 3$.
3. $-\frac{2c}{x^3}$.
4. $\frac{1}{2\sqrt{x}}$.
5. $6x + 9$.
6. $\frac{4}{(x+2)^2}$.
7. $2(x - 4)$.
8. $b + 2cx + 3dx^2$.
9. $a + gt$.
10. $3(x + 2)^2$.
11. nx^{n-1}.

Art. 19. Page 48.

1. $3x^2$.
2. $9x^2 - 12x + 7$.
3. $4x^3 - 3x^2 - 8x + 4$.
4. $3x^2 - 6x + 4$.
5. $8x^3 + 15x^2 - 6x - 5$.
6. $18x^2 + 6$.
9. $80 - 32t$; -16.

Art. 19. Page 50.

1. $-\frac{x+10}{x^3}$.
2. $\frac{x^4 - 9x^2}{(x^2-3)^2}$.
3. $-\frac{2}{x^3}$.
4. $-\frac{6}{(x-4)^2}$.
5. $\frac{3(1-t^2)}{(t^2+1)^2}$.
6. $\frac{1-3t^2}{(t^3-t+1)^2}$.
7. $1 + \frac{2}{\theta^3}$.
8. $-\frac{\theta^2+4}{(\theta^2-4)^2}$.
9. $\frac{6x^2 - 52x + 42}{(x^2-7)^2}$.
10. $\frac{-x^4 + 4x^3 - 16x^2 + 16x - 11}{(x^2-x+8)^2}$.
11. $\frac{-x^2 + 16x + 11}{(x^2+2x-5)^2}$.

Art. 20. Page 52.

1. $\frac{1}{2}x^{-\frac{1}{2}}$.
2. $\frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{1}{2}}$.
3. $-5x^{-6} - 6x^{-4} + 4x^{-2}$.
4. $9(3x - 7)^2$.
5. $2(2x^2 - 5x + 3)(4x - 5)$.
6. $-\frac{x}{\sqrt{1-x^2}}$.
7. $\frac{2x}{3(a^2+x^2)^{\frac{2}{3}}}$.
8. $-\frac{3}{2(3x-7)^{\frac{3}{2}}}$.
9. $\frac{6x^2}{(6-x^3)^3}$.
10. $2\theta + \frac{k}{2\theta^{\frac{1}{2}}}$.
11. $-\frac{Cn}{v^{n+1}}$.
12. $-\frac{1}{x^2} + \frac{6}{x^3} + \frac{24}{x^4}$.
13. $-\frac{3}{2(x-5)^{\frac{3}{2}}}$.
14. $\frac{-2x}{(1+x^2)\sqrt{1-x^4}}$.
15. $\frac{a-3x}{2\sqrt{a-x}}$.
16. $x^{p-1}(1-x)^{q-1}(p - px - qx)$.

Art. 21. Pages 53–54.

1. $5x^4 - 3 - 6x^{-4}$.

2. $\dfrac{x}{\sqrt{x^2-3}}$.

3. $\dfrac{2x-1}{2\sqrt{x^2-x+1}}$.

4. $-\frac{1}{2}\sqrt{\dfrac{x^3+1}{x^2-3}}\cdot\dfrac{x^4-9x^2-2x}{(x^3+1)^2}$.

5. $\frac{5}{2}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}$.

6. $\dfrac{(5x^2+2bx-3a^2)\sqrt{x+b}}{2\sqrt{x^2-a^2}}$.

7. $\dfrac{(x^2+9a^2)\sqrt{x}}{2(x^2+3a^2)^{\frac{3}{2}}}$.

8. $\dfrac{4x-x^2}{\sqrt[3]{(6x^2-x^3)^2}}$.

9. $\dfrac{(3a-x)\sqrt{2ax-x^2}}{(2a-x)^2}$.

10. $\dfrac{a+b+2x}{2\sqrt{(a+x)(b+x)}}$.

11. $\dfrac{x}{(2ax-x^2)^{\frac{3}{2}}}$.

12. $c-\dfrac{a}{t^2}$.

13. $\dfrac{6-3t^2}{2(2+t^2)^2\sqrt{t}}$.

14. $\dfrac{2\theta-\theta^2}{(1-\theta)^2}$.

15. $\dfrac{1-3\theta^2}{2(1+\theta^2)^2\sqrt{\theta}}$.

16. $50x^9 - 32x^7 + 70x^6 + 6x^5 - 40x^4 + 20x^3 + 6x^2 - 8x$.

17. $\dfrac{5(x^2-4)^{\frac{1}{3}}(10x^2+45x-4)}{3(x+5)^{\frac{2}{3}}}$.

18. $-\dfrac{2x+3\sqrt{cx}+2c}{8(x\sqrt{c}+x\sqrt{x})^{\frac{3}{2}}}$.

19. $\dfrac{(aq-bp)x^2+(ar-cp)x+(br-cq)}{(px^2+2qx+r)^{\frac{3}{2}}(ax^2+2bx+c)^{\frac{1}{2}}}$.

20. $-\dfrac{x^2+2a\sqrt{a^2-x^2}-2a^2}{x^2\sqrt{a^2-x^2}}$.

25. 2; 4.

26. (2, − 16); (− 2, 16).

Art. 22. Page 55.

1. $\dfrac{1}{3(x-4)^{\frac{2}{3}}}$.

2. $\dfrac{6x}{(1-x^2)^2}$.

3. $\dfrac{a}{2\sqrt{a\theta+b}}$.

4. $\sqrt{\dfrac{g}{2s}}$.

Art. 23. Page 55.

1. $6t-1$.

2. $\dfrac{2t^3-3t}{t-1}$.

3. $\dfrac{1}{4t^{\frac{3}{4}}}$.

4. $\dfrac{b-ct}{a}$; $\dfrac{a}{b-ct}$.

Miscellaneous Exercises. Pages 56–57.

1. $12x^3 - 12x + 7.$

2. $8x^7 - 7x^6 - 24x^5 + 20x^4.$

3. $10t - 3.$

4. $\frac{2\theta^3 - 1}{\theta^3}.$

5. $v_0 + gt.$

6. $\frac{9x^3 - 30x}{\sqrt{x^2 - 5}}.$

7. $2x(x-a)^3(x+b)^2(x-c) + 3x^2(x-a)^2(x+b)^2(x-c) + 2x^2(x-a)^3(x+b)(x-c) + x^2(x-a)^3(x+b)^2.$

8. $\frac{3cx^2\sqrt[n]{1+cx^3}}{n(1+cx^3)}.$

9. $-\frac{b}{x^3} - \frac{2c}{x^5}.$

10. $\frac{ad - bc}{(cx+d)^2}.$

11. $-\frac{nx^{n-1}}{(x^n + c^n)^2}.$

12. $\frac{1}{3}(ax^2 + bx + c)^{-\frac{2}{3}}(2ax + b).$

13. $\frac{5x - 3}{\sqrt{5x^2 - 6x + 12}}.$

14. $\frac{nx^{n-1}}{(1+x)^{n+1}}.$

15. $-\frac{nx + mx + an}{x^{n+1}(a+x)^{m+1}}.$

16. $\frac{(a+b-c-d)x^2 - 2(ab-cd)x - (a+b)cd + (c+d)ab}{(x-c)^2(x-d)^2}.$

17. $\frac{n(x + \sqrt{x^2-1})^n}{\sqrt{x^2-1}}.$

18. $\frac{\sqrt{x + \sqrt{1+x^2}}}{2\sqrt{1+x^2}}.$

19. $\frac{1}{(x+1)\sqrt{x^2-1}}.$

20. $\frac{x^2 - 6x - 5}{2(x-3)^{\frac{3}{2}}\sqrt{x^2+5}}.$

21. $-\frac{(2+x)\sqrt{1-x^2}}{(1+x)^2}.$

22. $2x + \frac{2x^3}{\sqrt{x^4-1}}.$

23. $\frac{8x^4 - 9a^2x^2 + 3ax^3}{3(x+a)(x^2-a^2)^{\frac{2}{3}}}.$

24. $2x - \frac{2x^2-1}{\sqrt{x^2-1}}.$

25. $-\frac{(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{1}{2}}}{x^{\frac{1}{3}}}.$

26. $\frac{1}{2}\sqrt{\frac{a}{\theta}} - \frac{3\sqrt{b\theta}}{2}.$

27. $\frac{a(c - 2d\theta)}{(1 - c\theta + d\theta^2)^2}.$

28. $\frac{7\sqrt{v} + 4}{12\sqrt[3]{v^2}\sqrt{\sqrt{v}+1}}.$

29. $4x - \frac{2(2x^2+1)}{\sqrt{x^2+1}}.$

30. $\frac{2a}{v^3} - \frac{c}{(v-b)^2}.$

31. 0; 12.

32. arc tan $(\pm \frac{3}{4}).$

33. $\frac{-3(u^2 + 10u + 15)x^3}{(u^2 - 15)^2}.$

34. $\frac{a}{b+v}.$

35. $-\frac{c}{v^2};\quad -\frac{c}{p^2}.$

36. (a) $-\frac{RT}{(v-a)^2} + \frac{2c}{T(v+b)^3}.$

(b) $\frac{R}{v-a} + \frac{c}{T^2(v+b)^2}.$

37. $1 + 0.00004\,\tau + 0.0000009\,\tau^2$; 1.0025025.

38. $-\dfrac{nC}{v^{n+1}}$; $-\dfrac{\sqrt[n]{C}}{np\sqrt[n]{p}}$.

39. 0; $\pm\dfrac{\sqrt{3}}{3}$; $\pm\dfrac{3\sqrt{7}}{7}$.

40. $-\dfrac{1}{s}\sqrt{\dfrac{ac}{2\,as - 2\,s^2}}$.

41. $\dfrac{x^2}{(ax+b)^2}$.

43. (a) x^3. (b) $x^4 - x$.

(c) $\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2\,x$. (d) $\dfrac{1}{x}$.

Art. 26. Page 65.

1. $x - y + 2 = 0$; $x + y = 6$.

2. $x_1y + y_1x = 2\,a^2$; $y - y_1 = \dfrac{x_1}{y_1}(x - x_1)$.

3. $2\,x - y = a$; $x + 2\,y = 3\,a$.

4. $3\,x + 4\,y = 25$; $4\,x - 3\,y = 25$.

5. $3\,y_1^2y = 2\,ax_1\,(x - x_1)$; $y - y_1 = -\dfrac{3\,y_1^2}{2\,ax_1}(x - x_1)$.

6. $y - y_1 = -\dfrac{y_1^{\frac{1}{3}}}{x_1^{\frac{1}{3}}}(x - x_1)$; $y - y_1 = \dfrac{x_1^{\frac{1}{3}}}{y_1^{\frac{1}{3}}}(x - x_1)$.

7. $yy_1^{n-1} + xx_1^{n-1} = a^n$; $y - y_1 = \left(\dfrac{y_1}{x_1}\right)^{n-1}(x - x_1)$.

8. $\dfrac{xx_1}{a^2} = \dfrac{yy_1}{b^2} = 1$; $y - y_1 = -\dfrac{a^2y_1}{b^2x_1}(x - x_1)$.

9. $y - y_1 = \dfrac{2\,x_1^3 + 1}{x_1^2}(x - x_1)$; $(y - y_1) = -\dfrac{x_1^2}{2\,x_1^3 + 1}(x - x_1)$.

10. $y - y_1 = (3\,x_1^2 - 3)(x - x_1)$; $y - y_1 = -\dfrac{1}{3\,x_1^2 - 3}(x - x_1)$.

Art. 27. Page 67.

1. 4; 4; $4\sqrt{2}$; $4\sqrt{2}$.

2. x_1; $\dfrac{y_1^2}{x_1}$; $\sqrt{x_1^2 + y_1^2}$; $\dfrac{y_1\sqrt{x_1^2 + y_1^2}}{x_1}$.

3. $\frac{4}{3}$; $\frac{3}{4}$; $\frac{5}{3}$; $\frac{5}{4}$.

4. $\dfrac{y_1}{3\,(x_1^2 - 1)}$; $3\,y_1\,(x_1^2 - 1)$; $\dfrac{y_1}{3\,(x_1^2 - 1)}\sqrt{9\,x_1^4 - 18\,x_1^2 + 10}$;
$y_1\sqrt{9\,x_1^4 - 18\,x_1^2 + 10}$.

5. $\frac{3}{2}\,a$; $2\,a$; $a\sqrt{13}$; $\dfrac{a}{3}\sqrt{13}$.

6. $\dfrac{a}{2}$; $2\,a$; $\dfrac{a}{2}\sqrt{5}$; $a\sqrt{5}$.

7. $x_1^{\frac{1}{3}}y_1^{\frac{2}{3}}$; $\left(\dfrac{y_1^4}{x_1}\right)^{\frac{1}{3}}$; $a^{\frac{1}{3}}\,y_1^{\frac{2}{3}}$; $\dfrac{a^{\frac{1}{3}}y_1}{x_1^{\frac{1}{3}}}$.

Miscellaneous Exercises. Pages 56–57.

1. $12x^3 - 12x + 7.$

2. $8x^7 - 7x^6 - 24x^5 + 20x^4.$

3. $10t - 3.$

4. $\dfrac{2\theta^3 - 1}{\theta^2}.$

5. $v_0 + gt.$

6. $\dfrac{9x^3 - 30x}{\sqrt{x^2 - 5}}.$

7. $2x(x-a)^3(x+b)^2(x-c) + 3x^2(x-a)^2(x+b)^2(x-c)$
$+ 2x^2(x-a)^3(x+b)(x-c) + x^2(x-a)^3(x+b)^2.$

8. $\dfrac{3cx^2\sqrt[n]{1+cx^3}}{n(1+cx^3)}.$

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24. $2x - \dfrac{2x^2 - 1}{\sqrt{x^2-1}}.$

25. $-\dfrac{(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{1}{2}}}{x^{\frac{1}{3}}}.$

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27. $\dfrac{a(c - 2d\theta)}{(1 - c\theta + d\theta^2)^2}.$

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29. $4x - \dfrac{2(2x^2+1)}{\sqrt{x^2+1}}.$

30. $\dfrac{2a}{v^3} - \dfrac{c}{(v-b)^2}.$

31. 0; 12.

32. arc tan $(\pm \frac{3}{4})$.

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37. $1 + 0.00004\,\tau + 0.0000009\,\tau^2$; 1.0025025.

38. $-\dfrac{nC}{v^{n+1}}$; $-\dfrac{\sqrt[n]{C}}{np\sqrt[n]{p}}$.

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41. $\dfrac{x^2}{(ax+b)^2}$.

43. (a) x^3. (b) $x^4 - x$.

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Art. 26. Page 65.

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4. $3\,x + 4\,y = 25$; $4\,x - 3\,y = 25$.

5. $3\,y_1^2y = 2\,ax_1(x - x_1)$; $y - y_1 = -\dfrac{3\,y_1^2}{2\,ax_1}(x - x_1)$.

6. $y - y_1 = -\dfrac{y_1^{\frac{1}{3}}}{x_1^{\frac{1}{3}}}(x - x_1)$; $y - y_1 = \dfrac{x_1^{\frac{1}{3}}}{y_1^{\frac{1}{3}}}(x - x_1)$.

7. $yy_1^{n-1} + xx_1^{n-1} = a^n$; $y - y_1 = \left(\dfrac{y_1}{x_1}\right)^{n-1}(x - x_1)$.

8. $\dfrac{xx_1}{a^2} = \dfrac{yy_1}{b^2} = 1$; $y - y_1 = -\dfrac{a^2y_1}{b^2x_1}(x - x_1)$.

9. $y - y_1 = \dfrac{2\,x_1^3 + 1}{x_1^2}(x - x_1)$; $(y - y_1) = -\dfrac{x_1^2}{2\,x_1^3 + 1}(x - x_1)$.

10. $y - y_1 = (3\,x_1^2 - 3)(x - x_1)$; $y - y_1 = -\dfrac{1}{3\,x_1^2 - 3}(x - x_1)$.

Art. 27. Page 67.

1. 4; 4; $4\sqrt{2}$; $4\sqrt{2}$.

2. x_1; $\dfrac{y_1^2}{x_1}$; $\sqrt{x_1^2 + y_1^2}$; $\dfrac{y_1\sqrt{x_1^2 + y_1^2}}{x_1}$.

3. $\frac{4}{3}$; $\frac{3}{4}$; $\frac{5}{3}$; $\frac{5}{4}$.

4. $\dfrac{y_1}{3\,(x_1^2 - 1)}$; $3\,y_1(x_1^2 - 1)$; $\dfrac{y_1}{3\,(x_1^2 - 1)}\sqrt{9\,x_1^4 - 18\,x_1^2 + 10}$;
$y_1\sqrt{9\,x_1^4 - 18\,x_1^2 + 10}$.

5. $\frac{3}{2}\,a$; $2\,a$; $a\sqrt{13}$; $\dfrac{a}{3}\sqrt{13}$.

6. $\dfrac{a}{2}$; $2\,a$; $\dfrac{a}{2}\sqrt{5}$; $a\sqrt{5}$.

7. $x_1^{\frac{1}{3}}y_1^{\frac{2}{3}}$; $\left(\dfrac{y_1^4}{x_1}\right)^{\frac{1}{3}}$; $a^{\frac{1}{3}}\,y_1^{\frac{2}{3}}$; $\dfrac{a^{\frac{1}{3}}y_1}{x_1^{\frac{1}{3}}}$.

Art. 29. Page 70.

1. $a\theta\sqrt{1+\theta^2}$; $a\theta^4$; $a\sqrt{1+\theta^2}$; a.

2. (a) $\frac{a\theta^2\sqrt{4+\theta^2}}{2}$; $\frac{a\theta^3}{2}$; $a\theta\sqrt{4+\theta^2}$; $2a\theta$.

(b) $\frac{(\theta+1)\sqrt{(\theta^2+\theta)^2+1}}{\theta}$; $(\theta+1)^2$; $\frac{\sqrt{(\theta^2+\theta)^2+1}}{\theta^2}$; $\frac{1}{\theta^2}$.

(c) $\frac{\rho\sqrt{4\rho^4+(a+2b\theta)^2}}{a+2b\theta}$; $\frac{2\rho^3}{a+2b\theta}$; $\frac{\sqrt{4\rho^4+(a+2b\theta)^2}}{2\rho}$; $\frac{a+2b\theta}{2\rho}$.

3. (a) $\frac{\theta}{2}$; (b) $(\theta^2+\theta)$; (c) $2\theta\frac{a+b\theta}{a+2b\theta}$.

Art. 30. Page 71.

1. (a) 2 is a double root. (b) 1 is a quadruple root. (c) 2 is a double root.

2. $4p^3+27r=0$, or $r=0$.

Art. 33. Page 75.

1. 0.16 in. 2. $\frac{2\Delta r}{r}$. 3. (b) $\frac{0.2}{i}$ per cent.

Art. 36. Pages 80–81.

1. (a) $v=v_0-gt$; $a=-g$.
   (b) 123.4 ft./sec.; -32.2 ft./sec.^{2}.
   (c) -102 ft./sec.; -32.2 ft./sec.^{2}.

2. $13\frac{107}{161}$ sec.; 751.5 ft.

8. 40 rad./sec.; 10 rad./sec.^{2}.

9. (a) 32 rad./sec.; -32 rad./sec.^{2}.
   (b) $3\frac{1}{2}$ sec.

10. (a) $\omega=a-3bt^2$; $\alpha=-6bt$.
    (b) $\omega=\frac{a}{2\sqrt{t}}$; $\alpha=-\frac{a\sqrt{t}}{4t^2}$.
    (c) $\omega=b+2ct$; $\alpha=2c$.

Art. 37. Page 83.

1. 0.54084; 0.2344; 0.23848. 2. -0.000067040; 0.00026816.

4. $R_0(\alpha+2\beta\tau)$.

Miscellaneous Exercises. Pages 83–85.

1. (a) 0; -1. (b) (0, 0); $(4a, -\frac{17}{3}a)$. (c) $x=2a\pm 2a\sqrt{2}$.

2. $\pm 45°$.

4. (a) $\begin{cases}\text{(a) } 12a^3x-y=20a^4; & 12a^3y+x=2a+48a^7.\\ \text{(b) } x+2y=2; & 4x-2y=3.\\ \text{(c) } 5y-2x=5; & 2y+5x=31.\end{cases}$

(b) $\begin{cases}\text{(a) } \frac{1}{3}a\sqrt{144a^6+1}; & 4a^4\sqrt{144a^6+1}; & \frac{1}{3}a; & 48a^7.\\ \text{(b) } \frac{1}{2}\sqrt{5}; & \frac{1}{4}\sqrt{5}; & 1; & \frac{1}{4}.\\ \text{(c) } \frac{3}{2}\sqrt{29}; & \frac{2}{5}\sqrt{29}; & \frac{15}{2}; & \frac{6}{5}.\end{cases}$

6. (a) $\rho=k\theta+c$. (b) $\theta+\frac{k}{\rho}=c$.

7. (a) 2 is a double root.
   (b) No multiple roots.
   (c) 4 is a double root.

9. $\frac{1}{2}$; $\frac{1}{2}$.

14. $\frac{pv^2+a}{v(v-b)}-\frac{2a}{v^2}$.

20. 0.825; 0.805.

Art. 40. Page 88.

5. $-\frac{a}{2x^2}-\frac{b}{x}+C.$

6. $\frac{5}{7}x^7-x^4+x^2+7x+C.$

7. $-\frac{3}{x}+4x-\frac{x^3}{3}+\frac{2x^5}{5}+C$

8. $-\frac{1}{x}-\frac{1}{3x^3}-\frac{1}{5x^5}+C.$

9. $\frac{3}{5}x^{\frac{5}{3}}-2x^{\frac{3}{2}}+C.$

10. $-2x^{-\frac{1}{2}}-4x^{\frac{1}{2}}+C.$

11. $-\frac{1}{2x^2}+\frac{4}{x}+C.$

12. $\frac{(x-1)^4}{4}+C.$

13. $\frac{16x^9}{9}-12x^8+\frac{216x^7}{7}-36x^6+\frac{81x^5}{5}+C.$

14. $\frac{(ax+b)^3}{3a}+C.$

15. $\frac{(x^2+4)^3}{3}+C.$

16. $\frac{(2x^4-5)^4}{32}+C.$

17. $\frac{(3x^2-7)^5}{30}+C.$

18. $2\sqrt{x^2-5x+7}+C.$

19. $\frac{3}{4}(x^2-6x+1)^{\frac{4}{3}}+C.$

20. $\frac{5m\sqrt{1+k^2x^2}}{k^2}+C.$

Art. 42. Pages 90–91.

1. $y=3x+C;\quad y=\frac{1}{6}x^3+C;\quad y=\frac{1}{2}mx^2+C;\quad y=\frac{1}{3}ax^3-bx+C;$

$$y=-\frac{1}{2ax^2}+C.$$

2. $y=x^3+C.$

3. $y=bx+C.$

4. $y^2=mx+C.$

5. $\theta=k\rho+C.$

6. $m\theta=\rho+k.$

7. $y=\frac{2}{3}x^3+5x-13.$

Art. 44. Page 93.

1. 6.5 ft./sec.; 384 ft.

2. $a=2t-6;\quad s=\frac{1}{3}t^3-3t^2.$

3. 786.1 ft.; $13\frac{137}{161}$ sec.

4. (a) $67\frac{1}{2}$ rad./sec. (b) $\frac{1525}{12\pi}$. (c) $\frac{1309}{2\pi}$.

Miscellaneous Exercises. Page 94.

1. (a) $\frac{1}{15}(3x-5)^5+C.$
   (b) $\frac{1}{12}(x^3-4)^4+C.$
   (c) $\frac{2}{9}(x^3-4)^{\frac{3}{2}}+C.$
   (d) $\frac{1}{9}(3x^2-5)^{\frac{3}{2}}+C.$

2. $3y=12x-x^3-15.$

3. $y^2=2ax+C.$

4. (a) $y=\frac{5}{2}x^2-4x+C.$
   (b) $y=x^3-\frac{1}{2}x^2+2x+C.$
   (c) $y=\frac{3}{4}x^{\frac{4}{3}}+C.$
   (d) $y=\frac{2}{3}x^{\frac{3}{2}}+2x+C.$

5. (a) $\frac{w(x-L)^3+wL^3}{6}$;
   (b) $\frac{L^2x^2}{2}-\frac{Lx^3}{3}+\frac{x^4}{12}.$
   (c) $\frac{wL^2x^2}{16}-\frac{wx^4}{24}.$
   (d) $\frac{3wLx^2}{16}-\frac{wx^3}{6}-\frac{wL^3}{48}.$

6. (a) $a + \frac{2b}{t^3}$; $\frac{at^2}{2} + \frac{b}{t} + c$. (b) $\frac{c}{2\sqrt{t}}$; $\frac{2}{3}ct^{\frac{3}{2}} + K$.
(c) $3t^2 - 5$; $\frac{1}{4}t^4 - \frac{5}{2}t^2 + 4t + C$. 7. $a\tau + \frac{1}{2}b\tau^2 + \frac{1}{3}c\tau^3$.

Art. 48. Page 102.

1. $dy = 5x^4\,dx$.
2. $dy = 6x\,dx$.
3. $dy = (9x^2 - 10x + 6)\,dx$.
4. $dy = (4x^3 - 18x^2 + 22x - 6)\,dx$.
5. $dy = \frac{5\,dx}{x^6}$.
6. $dy = -\frac{2\,dx}{(x-5)^2}$.
7. $dy = \frac{(a^4 - 2a^2x^2 - x^4)\,dx}{(a^2 + x^2)\sqrt{a^4 - x^4}}$.
8. $dy = \frac{(8x - 5a)\,dx}{15x^{\frac{2}{3}}(x-a)^{\frac{4}{5}}}$.
9. $dy = (6x^5 - 12x^3 + 10x)\,dx$.
10. $dy = \frac{(9x^2 + 4)\,dx}{3(3x^3 + 4x - 8)^{\frac{2}{3}}}$.
11. $dr = \frac{3}{5}\theta^2 d\theta$.
12. $ds = (2at + b)\,dt$.
13. $dQ = (b + 2c\tau)\,d\tau$.
14. $du = -\left(\frac{C}{T^2} + \frac{2D}{T^3}\right)dT$.
15. $dp = -\frac{mc\,dv}{v^{m+1}}$.
16. $dy = bmn(a + bx^n)^{m-1}x^{n-1}dx$.

Art. 49. Pages 103–104.

1. (a) $\frac{2xy}{3y^2 - x^2}$. (b) $-\frac{3x^2y + 2xy^2 + 4y^3}{x^3 + 2x^2y + 12xy^2 - 4y^3}$. (c) $\frac{b^2x}{a^2y}$.
(d) $-\frac{y+d}{x+c}$. (e) $-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$.

2. $-\frac{ax + hy + g}{hx + by + f}$. 3. $\frac{1}{4}$. 4. $-\frac{2}{7}$; $\frac{2}{7}$. 5. $\frac{1}{3}$.

6. (a) Subtan $= 4\rho^2\theta^{\frac{1}{2}} - 2\rho\theta$; subn $= \frac{\rho}{4\rho\theta^{\frac{1}{2}} - 2\theta}$.
(b) Subtan $= 1 - 2\rho\theta$; subn $= \frac{\rho^2}{1 - 2\rho\theta}$.

7. (a) $-\frac{np}{mv}$. (b) $\frac{pv^3 - av + 2ab}{(b-v)v^3}$.

8. (a) $\frac{2(1+x)y + 2x + y^2}{2y(1-x) - (1+x)^2}$. (b) $-\frac{y^2 + b}{2xy}$. (c) $\frac{5x^4 - 12x^2y + 2xy^3}{3x^4 - 3x^2y^2 + 20y^4}$

Art. 50. Pages 105–106.

1. $\frac{24}{5}$; $\frac{8\sqrt{34}}{5}$.
2. $-12\sqrt{5}$; $6\sqrt{5}$.
3. (a) 6 ft./sec.; (b) 10 ft./sec.
4. (a) $\theta = a\rho$. (b) $\rho^2 = k\theta$.
5. 2π cu. in. /sec.
6. $3\frac{3}{4}$ mi./hr.
7. $\frac{\pi}{\sqrt{gL}}$.
8. $x^3 - ky = 0$.
9. 6; 5.
10. $\frac{80.64}{\pi}$ ft./sec.

Art. 51. Page 107.

1. $\frac{1}{4}x^4 + C.$

2. $\frac{4}{3}x^3 - 3x^2 + C.$

3. $\frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x + C.$

4. $-\frac{1}{x} + C.$

5. $\frac{5}{2}x^{-2} + C.$

6. $-\frac{3}{x} + \frac{7}{2x^2} + C.$

7. $-\frac{6}{5x} + \frac{1}{5x^2} - \frac{1}{5x^3} + C.$

8. $\frac{1}{3}ax^3 + bx - \frac{c}{x} + C.$

9. $-\frac{5}{2x^2} + \frac{8}{x} + 3x + C.$

10. $at + \frac{1}{2}bt^2 + \frac{1}{3}ct^3 + C.$

11. $\frac{\theta^2}{2} - \frac{\theta^4}{12} + \frac{\theta^6}{30} - \frac{\theta^8}{56} + C.$

Miscellaneous Exercises. Pages 107-109.

1. (a) $\frac{2nx^{2n-1}\,dx}{(1+x^2)^{n+1}}.$ (b) $-\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{x^2\sqrt{1-x^4}}\,dx.$

(c) $\frac{5}{3}\left[\frac{6x^3+35x^2-12x-56}{x^{\frac{1}{3}}(x^2-4)^{\frac{1}{2}}(x+7)^{\frac{2}{3}}}\right]dx.$

2. (a) $\frac{3x^2-10axy}{5ax^2-21y^2}.$ (b) $\frac{2x(1-y^2)^{\frac{3}{2}}-3y(1-y^2)}{2y(1-y^2)^{\frac{3}{2}}+3x}.$

(c) $-\frac{a(ax+by+c)-x}{b(ax+by+c)-y}.$ (d) $\frac{xy^2}{(b+y)(a^2-by-2y^2)-x^2y}.$

3. (a) $\frac{1}{6}(x^3-5)^2 + C.$ (b) $\frac{1}{9}(3x^2-7)^{\frac{3}{2}} + C.$ (c) $4\sqrt{x^2+5x+3} + C.$

(d) $-\frac{2}{3}x^{-\frac{3}{2}} + 6x^{-\frac{1}{2}} + 4x^{\frac{1}{2}} - \frac{10}{3}x^{\frac{3}{2}} + C.$

5. $v_x = \frac{12y}{\sqrt{100+y^2}};\quad v_y = \frac{120}{\sqrt{100+y^2}}.\quad 6\sqrt{2};\quad 6\sqrt{2}.$

6. 4 ft./sec.; 3 ft./sec.

7. $\frac{4}{3}\sqrt{11}$ ft./sec.; $6\frac{2}{3}$ ft./sec.

8. $17\frac{7}{9}$ ft./sec.; $\frac{5}{9}\sqrt{1105}$ ft./sec.

9. $14\sqrt{3}$ ft./sec.; -14 ft./sec.

10. $\frac{C\tan\frac{1}{2}\beta}{\pi a^2}$ ft./sec.

11. $\frac{\sqrt{16.1}}{50}$ ft./sec. per sec.

15. -2.24 lb./sq. in. per sec.

16. $\frac{0.8}{3}\sqrt{3}$ in./min.

17. $\frac{T_0}{T^2}.$

18. $0.1\,R_0(a + 2b\tau).$

19. $\frac{R_0(c-2d\tau_1)}{(1-c\tau_1+d\tau_1^2)^2}.$

20. (a) $\pm\frac{a}{4}\sqrt{6}.$ (b) $0.899\,a.$

(c) $y - \frac{a}{2}\sqrt{2\sqrt{3}-3} = -(3+2\sqrt{3})\sqrt{2\sqrt{3}-3}\left(x - \frac{a}{2}\right).$

Art. 52. Page 113.

1. $a \cos ax$.

2. $-3 \sin x \cos^2 x$.

3. $2 (\cos 2x - \sin 2x)$.

4. $2 \tan x \sec^2 x$.

5. $a \cos ax \cos bx - b \sin ax \sin bx$.

6. $\sin^3 x (3 + \sec^2 x)$.

7. $\frac{1}{2} \cos \frac{x}{2}$.

8. $\frac{1}{2} \sec^2 \frac{x}{2}$.

9. $\frac{1}{2} x \sec^2 \frac{x}{2} + \tan \frac{x}{2}$.

10. $\tan^2 \theta$.

11. $-2 \sin 2x$.

12. $\frac{2 \cos x}{(1 - \sin x)^2}$.

13. $-2a \sin 2\theta$.

14. $b \sin \theta$.

15. $-\frac{\sin \theta}{(1 - \cos \theta)^2}$.

16. $\frac{2a (2 \sin \theta - \sin^3 \theta)}{\cos^2 \theta}$.

17. $-\frac{a \sin 2\theta}{\sqrt{\cos 2\theta}}$.

18. $a (\sin n\theta)^{\frac{1-n}{n}} \cos n\theta$.

19. $a \sec^3 \frac{\theta}{3} \tan \frac{\theta}{3}$.

20. $-\frac{a}{2}\sqrt{\frac{r}{t}} \sin \sqrt{rt}$.

21. $-\omega (A \sin \omega t + B \cos \omega t)$.

22. $r \sin \theta \left[1 + \frac{r \cos \theta}{\sqrt{L^2 - r^2 \sin^2 \theta}}\right]$.

23. $\frac{\sin \phi}{1 - \cos \phi}$.

24. $a\phi \cos \phi$; $a\phi \sin \phi$; $\tan \phi$.

25. (a) $\frac{\sin^2 \theta}{\cos \theta}$; $\cos \theta$.

 (b) $\frac{a (1 - \cos \theta)^2}{\sin \theta}$; $a \sin \theta$.

 (c) $a \sec^3 \frac{\theta}{2} \cot \frac{\theta}{2}$; $a \sec^2 \frac{\theta}{2} \tan \frac{\theta}{2}$.

27. $\tan x$; $a^2 \sin x \cos x$.

28. 90°.

29. $\text{arc} \tan (2\sqrt{2})$.

30. $\frac{\pi}{4} + n\pi$.

31. 0.00027; − 0.00011.

Art. 53. Page 117.

1. $\frac{1}{\sqrt{a^2 - x^2}}$.

2. $\frac{\sqrt{k}}{k + x^2}$.

3. $-\frac{2a^2}{x\sqrt{x^4 - a^4}}$.

4. $\frac{x}{1 + x^2} + \text{arc} \tan x$.

5. $-\frac{3}{\sqrt{25 - (3x - 2)^2}}$.

6. $\frac{2}{\sqrt{1 - x^2}}$.

7. 0. 8. 1.

9. $\frac{1}{\sqrt{1 - t^2}}$.

10. $\frac{\sqrt{2}}{(1 + t^2)\sqrt{1 - t^2}}$.

11. $-\frac{k^2 \sec^2 x}{2 (2 - k^2 \tan x)\sqrt{1 - k^2 \tan x}}$.

12. $\frac{6}{\sqrt{6a^2 - 9x^2}}$.

13. $-\frac{x}{(a^2 + x^2)(a - x)^2}$.

14. $\frac{1}{n (m^2 + x^2)}$.

Art. 55. Page 121.

1. $\frac{2x-3}{x^2-3x+5}$.

2. $-\frac{2m}{(x^2-m^2)\log a}$.

3. $\frac{xa^{\sqrt{x^2-a^2}}\log a}{\sqrt{x^2-a^2}}$.

4. $3x^2e^{x^3}+e^x$.

5. $1+\log x$.

6. $\frac{1}{x\log x}$.

7. e^x-e^{-x}.

8. $-\frac{4}{(e^x-e^{-x})^2}$

9. $e^{\sin x}\cos x$.

10. $\frac{\sec^2 x}{\tan x}$.

11. $-\frac{2}{\sqrt{x^2+1}}$.

12. $\frac{1-2x+2x^2-3x^3}{\sqrt{1+x^2}}$.

13. $\frac{3(x-5)^{\frac{1}{2}}(3x-4)^{\frac{4}{3}}}{2(2x+5)^{\frac{6}{5}}}+\frac{4(x-5)^{\frac{3}{2}}(3x-4)^{\frac{1}{3}}}{(2x+5)^{\frac{6}{5}}}-\frac{6(x-5)^{\frac{3}{2}}(3x-4)^{\frac{4}{3}}}{5(2x+5)^{\frac{11}{5}}}$

14. $\frac{(3x^2-1)(x^3+1)}{x^3(1-x^2)^{\frac{3}{2}}}$.

15. $-\frac{1}{(1+x)\sqrt{1-x^2}}$.

16. $\sqrt{k}\,(Ae^{\theta\sqrt{k}}-Be^{-\theta\sqrt{k}})$.

17. $e^{-kt}(B-Bkt-Ak)$.

18. $e^{-kt}[(ma-bk)\cos mt-(mb+ak)\sin mt]$.

19. $\frac{e^{a\theta}}{a}$; $ae^{a\theta}$; $\frac{e^{a\theta}\sqrt{a^2+1}}{a}$; $e^{a\theta}\sqrt{a^2+1}$.

20. 45°; $\arctan\frac{1}{e^2}$.

21. $y=e^{cx}+C$.

22. $\frac{a(e^{\frac{x}{a}}+e^{-\frac{x}{a}})}{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}$; $\frac{a(e^{\frac{2x}{a}}-e^{-\frac{2x}{a}})}{4}$; $\frac{a(e^{\frac{x}{a}}+e^{-\frac{x}{a}})^2}{4}$.

23. 1.4656.

Art. 56. Page 124.

4. $\sinh^3 x$.

5. $\tanh^3 x$.

6. (a) $y-y_1=\sinh\frac{x_1}{a}(x-x_1)$. (b) $a\coth\frac{x}{a}$.

Art. 58. Page 128.

1. $\frac{1}{5}(x+a)^5+C$.

2. $\frac{2}{3}(x^3-a^2)^{\frac{3}{2}}+C$.

3. $\log(3x^2+a^2)+C$.

4. $e^{x^2}+C$.

5. $-\frac{1}{\sin\theta}+C$.

6. $-\frac{1}{3}\cos^3\theta+C$.

7. $\frac{1}{2}(\arctan x)^2+C$.

8. $-\frac{1}{m}\sqrt{1-m^2x^2}+C$.

9. $\frac{1}{2}\tan^2\theta+C$.

11. $\frac{2}{7a}(ax+b)^{\frac{7}{2}}+C$.

12. $\frac{1}{6}\log(2x^3-5)+C$.

13. $\sqrt{a^2+x^2}+C$.

14. $\frac{1}{c}\arcsin cx+C$.

15. $\operatorname{arc\,sec} ax+C$.

16. $\frac{a^{m^2x^2}}{m^2\log a}+C$.

Art. 59. Pages 131-132.

1. $\dfrac{3\,mx^{c-1}}{c-1} + C.$

2. $-\dfrac{2}{3\sqrt{m^3 + x^3}} + C.$

3. $2\sqrt{5\,x - x^2} + C.$

4. $\sqrt{x^2 - 4\,x + 5} + C.$

5. $\frac{1}{2}\,e^{x^2} + C.$

6. $\frac{1}{4}\sin^4\theta + C.$

7. $-\dfrac{1}{3\sin^3\theta} + C.$

8. $\sin(x + a) + C.$

9. $3\arcsin\dfrac{x}{3} - 5\sqrt{9 - x^2} + C.$

10. $\dfrac{2\,(ax + b)^{\frac{5}{2}}}{5\,a} + C.$

11. $\dfrac{\log(4\,x^3 + 7)}{12} + C.$

12. $\log\sqrt{x^4 + 2\,x^3 - 4\,x^2 - 6\,x - 10} + C.$

13. $\log\left[x + 2 + \sqrt{x^2 + 4\,x + 13}\right] + C.$

14. $\frac{1}{3}\,x^3 - \frac{1}{2}\,x^2 + x - \log(x + 1) + C.$

15. $\log(x^3 + x) - \frac{1}{2}\log(x^2 + 1) + C.$

16. $-\dfrac{1}{b}\log(a + b\cos x) + C.$

17. $-e^{\cos x} + C.$

18. $\arctan x - \log\sqrt{1 + x^2} + C.$

19. $\log\dfrac{2\,x + 1}{(x + 1)} + C.$

20. $\arcsin\dfrac{2\,x + 3}{\sqrt{57}} + C.$

21. $\log\left[x + 3 + \sqrt{x^2 + 6\,x + 1}\right] + C.$

22. $\dfrac{1}{2\sqrt{6}}\log\dfrac{z - \sqrt{6}}{z + \sqrt{6}} + C.$

24. $\arcsin(\log x) + C.$

23. $-\dfrac{1}{4\sqrt{3}}\log\dfrac{\sqrt{3}\,\theta - 2}{\sqrt{3}\,\theta + 2} + C.$

25. $\log\left[s + a + \sqrt{s^2 + 2\,as}\right] + C.$

26. $c\log\left[u + m + \sqrt{(u + m)^2 - n^2}\right] + C.$

27. $\dfrac{\log\sec(mx + b)}{m} + C.$

28. $\log\dfrac{1}{\sqrt[b]{a + be^{-x}}} + C.$

29. $\log(\log x) + C.$

30. $\dfrac{1}{6}\arcsin\dfrac{x^6}{a^6} + C.$

31. $\log(e^x + \sin x + \cos x) + C.$

32. $\dfrac{1}{3}\arctan\left(\dfrac{2\,x + 1}{3}\right) + C.$

33. $-\cos(x^2 - 7x) + C.$

34. $\dfrac{2\,(x + 3)}{\sqrt{x}} + C.$

35. $\arcsin x + \sqrt{1 - x^2} + C.$

Art. 60. Page 134.

1. $y = a\sin\omega t\,;\ v = a\omega\cos\omega t\,;\ \alpha = -\,y\omega^2.$

3. $y = a\sin\left(\omega t + \dfrac{\pi}{6}\right);\ v_y = a\omega\cos\left(\omega t + \dfrac{\pi}{6}\right);\ a_y = -\,y\omega^2.$

$x = a\cos\left(\omega t + \dfrac{\pi}{6}\right);\ v_x = -\,a\omega\sin\left(\omega t + \dfrac{\pi}{6}\right);\ a_x = -\,x\omega^2.$

Miscellaneous Exercises. Pages 137-139.

4. (a) $\frac{\pi}{4} + n\pi$; $\pm \frac{\pi}{4}$; $\frac{3}{4}\pi + n\pi$. (b) $\frac{n\pi}{2}$; arc tan (± 2); $\frac{\pi}{4} + \frac{n\pi}{2}$.

(c) $\frac{3\pi}{4} + n\pi$; arc tan $(\pm \sqrt{2})$; $\frac{\pi}{4} + n\pi$. (d) $x = 1$; $\frac{\pi}{4}$; nowhere.

5. (a) 0.80902. (b) $-$ 0.58779. (c) 1.52786.

6. 15.7′. **7.** $\frac{\sin\theta}{1 - \cos\theta}$; 0.5773; π.

8. (a) $\frac{\sqrt{a^2 - x^2}}{x}$. (b) $\frac{x^2}{\sqrt{2ax + x^2}}$. (c) $\sqrt{a^2 - x^2}$.

(d) $\frac{1}{x^4(1 + x^2)}$. (e) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$. (f) $-\frac{a^{\frac{1}{x}} \sec^2 a^{\frac{1}{x}} \log a}{x^2}$.

9. (a) $\frac{a(1 + \cos\theta)^2}{\sin\theta}$; $a \sin\theta$. (b) $\frac{2\rho}{a}$; $\frac{a\rho}{2}$.

(c) $\frac{1}{2}\rho \tan\theta$; $a^2 \sin 2\theta$. (d) $\rho \cot\frac{\theta}{2}$; $\rho \tan\frac{\theta}{2}$.

10. (a) $-\frac{1 + \cos\theta}{\sin\theta}$. (b) $\frac{2}{a}$. (c) $\frac{\tan\theta}{2}$. (d) $\cot\frac{\theta}{2}$.

12. $m \cos\frac{\theta}{2}$; $m \sin\frac{\theta}{2}$. **13.** $\cot C \, \Delta C$.

14. (a) arc tan $e^x + C$.

(b) $\frac{2(e^x + 1)^{\frac{1}{2}}(e^x - 2)}{3} + C$.

(c) $\log\frac{e^x - 1}{e^x + 1} + C$.

(d) $\frac{1}{3}x^3 - x +$ arc tan $x + C$.

(e) $\frac{3}{2\sqrt{5}}$ arc tan $\frac{x^2}{\sqrt{5}} + C$.

(f) $\frac{5}{2}$ arc sec $\frac{3x}{2} + C$.

(g) $\frac{1}{b}$ arc sec $\frac{x - a}{b} + C$.

15. $\int \frac{dx}{\sqrt{x^2 + a^2}} =$ inv. sinh $\frac{x}{a} + C$; $\int \frac{dx}{\sqrt{x^2 - a^2}} = \pm$ inv. cosh $\frac{x}{a} + C$;

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \text{ inv. } \tanh\frac{x}{a} + C.$$

16. (a) $-p\left(\frac{B}{T^2} + \frac{2C}{T^3}\right)$. (b) $\frac{acb^{\frac{\tau}{c+\tau}} \log b}{(c + \tau)^2}$. (c) $p(b\alpha^\theta \log\alpha - c\beta^\theta \log\beta)$.

17. $v = \sqrt{\frac{g}{k}} \, \frac{e^{2t\sqrt{gk}} - 1}{e^{2t\sqrt{gk}} + 1}$. **18.** $\frac{C + e^{-kt}(kv_0 - C)}{k}$.

19. $-ae^{-\lambda t}\{2\pi b \sin 2\pi(bt + c) + \lambda \cos 2\pi(bt + c)\}$.

Art. 65. Page 144.

1. (a) $6x - 6$.

(b) $x^x \left[\frac{1}{x} + (\log x + 1)^2\right]$.

(c) $e^{ax}\,[2a\cos x + (a^2 - 1)\sin x]$.

(d) $\frac{x^2 - 2}{(1 - x^2)^{\frac{3}{2}}}$.

(e) $2\sec^3 x - \sec x$.

(f) $-\frac{2\theta}{(1 + \theta^2)^2}$.

2. (a) $\frac{2}{x}$.

(b) $6\sec^4 x - 4\sec^2 x$.

(c) $2\sec^2 x\,(x + 3\tan x + 3x\tan^2 x)$.

(d) $6 - 60\sin^2 x + 64\sin^4 x$.

3. (a) $12x^3 - 21x^2 + 5$; $36x^2 - 42x$; $72x - 42$; 72; 0.

(b) $(x + 2)^{-1}$; $-(x + 2)^{-2}$; $2(x + 2)^{-3}$; $-6(x + 2)^{-4}$; $24(x + 2)^{-5}$.

(c) $2x^3 + 8x^3\log x$; $14x^2 + 24x^2\log x$; $52x + 48x\log x$;

$100 + 48\log x$; $\frac{48}{x}$.

4. (a) $a^x(\log a)^n$.

(b) $\frac{(-1)^n\,n!}{x^{n+1}}$.

(c) $\frac{(-1)^{n-1}(n-1)!}{x^n}$.

5. (a) $-\frac{4p^2}{y^3}$.

(b) $-\frac{b^4}{a^2y^3}$.

6. (a) $\frac{12(x^7y + 7x^6y^2 + 23x^5y^3 + 38x^4y^4 + 23x^3y^5 + 7x^2y^6 + xy^7)}{(x^3 + 6x^2y + 3xy^2)^3}$.

(b) $\frac{(a^2 - 1)(y^2 - 2axy + x^2)}{(y - ax)^3}$.

(c) $-y$.

8. (a) $\frac{1}{a^3}\sinh\frac{\theta}{a}$.

(b) $2\csc^2\theta\cot\theta$.

(c) $\frac{3}{4}\csc^2\frac{\theta}{2}\left(3\csc^2\frac{\theta}{2} - 2\right) - \frac{1}{2}\theta\csc^2\frac{\theta}{2}\cot\frac{\theta}{2}\left(3\csc^2\frac{\theta}{2} - 1\right)$.

(d) $an^3e^{n\theta}$.

Art. 66. Page 146.

1. $\frac{\sin mx}{m} + C_1$; $-\frac{y}{m^2} + C_1x + C_2$;

$-\frac{\sin mx}{m^3} + \frac{C_1x^2}{2} + C_2x + C_3$; $\frac{y}{m^4} + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4$.

2. $\frac{1}{3}x^3 + x + C_1$; $\frac{1}{12}x^4 + \frac{1}{2}x_2 + C_1x + C_2$; $\frac{1}{60}x^5 + \frac{1}{6}x^3 + \frac{1}{2}C_1x^2 + C_2x + C_3$; $\frac{1}{360}x^6 + \frac{1}{24}x^4 + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$.

3. $-\frac{1}{2x^2} + C_1$; $\frac{1}{2x} + C_1x + C_2$; $\frac{1}{2}\log x + \frac{C_1x^2}{2} + C_2x + C_3$.

4. $y = \frac{kx^3}{6} + \frac{6b - 6n - a^2k + m^2k}{6(a - m)}x + \frac{6an - 6bm + ka^2m - kam^2}{6(a - m)}$.

6. $C' = 0$; $C'' = 0$.

Art. 67. Pages 150-151.

1. Max. if $x = 0$; min. if $x = 1$.

2. Max. if $x = \frac{4}{3}$; min. if $x = 2$.

3. Max. if $x = 2$; min. if $x = 3$.

4. Max. if $x = -1$; min. if $x = 3$.

5. Min. if $x = 1$.

6. Min. if $x = \frac{1}{2m}\log\frac{a_2}{a_1}$.

7. Max. if $x = \pm 2n\pi$, min. if $x = \pi \pm 2n\pi$.

8. Min. if $x = \frac{1}{e}$.

9. Min. if $x = \frac{a^2}{a+b}$, max. if $x = \frac{a^2}{a-b}$.

10. No max. nor min.

11. No max. nor min.

12. Max. if $x = 0$.

13. Min. if $x = 2$.

14. Max. if $x = 1$; min. if $x = -1$.

Art. 68. Pages 152-155.

1. $\frac{m}{2}$ and $\frac{m}{2}$.

2. $\frac{a}{2}$ and $\frac{a}{2}$.

3. 45°.

4. $-\mu + \sqrt{1+\mu^2}$.

5. $\frac{a}{6}$.

6. 8 inches.

7. Length = diameter.

8. $\frac{1}{\sqrt{e}}$.

9. $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

10. Altitude = 2 × diameter of sphere.

12. (a) Base = altitude = $r\sqrt{2}$.
(b) $a\sqrt{2}$ and $b\sqrt{2}$. (c) base = $a\sqrt{3}$, altitude = $\frac{3}{2}a$.

13. (a) Altitude = $\frac{1}{3}$ × altitude of cone.
Radius of base = $\frac{2}{3}$ × radius of base of cone.
(b) Altitude = $\frac{4}{3}a$; radius of base = $\frac{2}{3}a\sqrt{2}$.
(c) Radius = $\frac{1}{2}\sqrt{2}$ × radius of base of paraboloid.
Altitude = $\frac{1}{2}$ × altitude of paraboloid.

14. (a) Altitude = $\frac{1}{3}\sqrt{3}$ × slant height.
Radius of base = $\frac{1}{3}\sqrt{6}$ × slant height.
(b) Radius of base = $\frac{1}{2}\sqrt{\frac{S}{\pi}}$. Altitude = $\sqrt{\frac{2S}{\pi}}$.

15. Radius of base = $\frac{\sqrt{2}}{2}\sqrt[3]{\frac{6V}{\pi}}$; altitude = $\sqrt[3]{\frac{6V}{\pi}}$.

16. $3b$. **17.** $\frac{c}{2}$. **18.** $\sqrt{\frac{gT}{3w}}$; 26.856. **19.** $\frac{25}{12}$. **20.** a.

21. Breadth = $\frac{14\sqrt{3}}{3}$ in.; depth = $\frac{14\sqrt{6}}{3}$ in.

22. Breadth = 6 in.; depth = $6\sqrt{3}$ in.

24. $\theta = \text{arc tan}\,(\pm\sqrt{2})$.

25. $p = \sqrt{p_1 p_a}$.

Art. 69. Page 156.

1. (a) Concave up if $x > \frac{2}{3}$; concave down if $x < \frac{2}{3}$.

(b) Concave up if $-\frac{\pi}{2} + 2\,n\pi < x < \frac{\pi}{2} + 2\,n\pi$;

Concave down if $\frac{\pi}{2} + 2\,n\pi < x < (2\,n + 1)\,\pi$.

(c) Concave up if $x < -1$; concave down if $x > -1$.

2. (a) Concave down at every point.
(b) Concave up at every point.

5. (a) Concave up at every point.
(b) Concave down at every point.

6. Concave up at every point.

Art. 70. Page 158.

2. (a) $x = -\frac{12}{9}$. (b) $x = 0$, or $-\frac{7}{10}$. (c) No point of inflexion.

3. $y = \frac{1}{3}x^3 + x + 3$.

4. (a) $x = 0 + 2\,n\pi$. (b) $x = 0$, or $\pm\sqrt{\frac{3}{2}}$. (c) $x = 0$.

5. $v = 0.2299$, or 0.00297.

Art. 73. Page 166.

1. $-\frac{(y_1^2 + a^2)^{\frac{3}{2}}}{a^2}$.

2. $\frac{(9\,a^2y_1^4 + 4\,x_1^2)^{\frac{3}{2}}}{3\,ay_1\,(6\,ay_1^3 - 8\,x_1^2)}$.

3. $-\frac{(a^4y_1^2 + b^4x_1^2)^{\frac{3}{2}}}{a^4b^4}$.

4. $-\frac{[(x_1 + a)^2 + 1]^{\frac{3}{2}}}{x_1 + a}$.

5. $\sec x_1$.

6. $\frac{[1 + \operatorname{sech} x_1 \tanh^2 x_1]^{\frac{3}{2}}}{\operatorname{sech} x_1 - 2\operatorname{sech}^3 x_1}$.

7. $\cosh^2 x_1$.

8. $3\,(ax_1y_1)^{\frac{1}{3}}$.

9. $\frac{y_1^2}{a}$.

10. $-\frac{(a^4y_1^2 + b^4x_1^2)^{\frac{3}{2}}}{a^4b^4}$.

11. $\frac{(4\,y_1^4 + x_1)^{\frac{3}{2}}\sqrt{x_1}}{2\,y_1^4 - 4\,x_1^{\frac{3}{2}}y_1}$.

12. $\frac{a\sqrt{x_1}\,(8\,a - 3\,x_1)^3}{3\,(2\,a - x_1)^2}$.

13. $\frac{(x_1 + y_1)^{\frac{3}{2}}}{c^{\frac{1}{2}}}$.

14. $a \sec \frac{x_1}{a}$.

15. $-15\sqrt{3}$; $(20, -10\sqrt{2})$.

16. $\frac{(109)^{\frac{3}{2}}}{60}$; $(21\frac{1}{6}, 15\frac{9}{20})$.

17. -1; $(0, 0)$.

18. $-\frac{a^2}{3\,b}$; $\left(0, b - \frac{a^2}{3\,b}\right)$.

19. a; $(a, 0)$.

Art. 74. Page 168.

1. $\dfrac{(\rho_1^2 + a^2)^{\frac{3}{2}}}{\rho_1^2 + 2a^2}$. 5. $\dfrac{\rho_1^3}{a^2}$.

2. $\dfrac{c\,(1 + \theta_1^2)^{\frac{3}{2}}}{\theta_1^4}$. 6. $\frac{2}{3}\sqrt{2a\rho_1}$.

3. $\dfrac{a\,(5 - 4\cos\theta_1)^{\frac{3}{2}}}{9 - 6\cos\theta_1}$. 7. $\rho_1\sqrt{1 + a^2}$.

4. $2a\sec^3\dfrac{\theta_1}{2}$. 8. $\dfrac{a^2}{b}(1 - e^2\cos^2\theta)^{\frac{3}{2}}$

Miscellaneous Exercises. Pages 169-171.

1. (a) $\dfrac{-6a^2x^5 + 54x^4y + 6a^3x^4y - 72ax^3y^2 + 12a^2x^2y^3 + 54xy^4 - 18ay^5}{(ax^2 - 3y^2)^3}$

(b) $2e^x\cos\left(x + \dfrac{\pi}{4}\right)$. (c) $\dfrac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}$.

(d) $\dfrac{a^2(4a + 2x)}{(x - a)^4}$. (e) $\sin x\,(2 - x^2) + 4x\cos x$.

5. (a) $4e^x(\sin x - \cos x)$. (b) $\dfrac{6}{x}$.

6. $v = \omega_0\left[r\sin\theta + \dfrac{r^2\sin 2\theta}{2\sqrt{L^2 - r^2\sin^2\theta}}\right]$;

$a = \omega_0^2\left[r\cos\theta + \dfrac{r^2L^2\cos 2\theta + r^4\sin^4\theta}{(L^2 - r^2\sin^2\theta)^{\frac{3}{2}}}\right]$.

7. $r\omega_0^2\left(1 + \dfrac{r}{L}\right), - r\omega_0^2\dfrac{r}{\sqrt{L^2 - r^2}}, r\omega_0^2\left(-1 + \dfrac{r}{L}\right)$.

8. $ax^2 - (2a - 3)x$. 9. $y = \dfrac{Px^2}{6EI}(x - 3L)$.

10. (a) $\frac{1}{3}x^3 + 3x + C_1$; $\frac{1}{12}x^4 + \frac{3}{2}x^2 + C_1x + C_2$;

$\frac{1}{60}x^5 + \frac{1}{2}x^3 + \frac{1}{2}C_1x^2 + C_2x + C_3$.

(b) $\dfrac{e^{mx}}{m} + C_1; \dfrac{e^{mx}}{m^2} + C_1x + C_2\ \dfrac{e^{mx}}{m^3} + \dfrac{C_1x^2}{2} + C_2x + C_3$.

(c) $-\cos\dfrac{(kt + \epsilon)}{k} + C_1$; $-\dfrac{\sin(kt + \epsilon)}{k^2} + C_1x + C_2$;

$\cos\dfrac{(kt + \epsilon)}{k^3} + \dfrac{C_1x^2}{2} + C_2x + C_3$.

11. 0.5266. 12. (a) $\frac{1}{2}c\cos\alpha$; $\cos^2\alpha$. (b) $c\cos\alpha$; $kc^2\cos^2\alpha$.

(c) $\sqrt{\dfrac{gh}{\sqrt{1-c^2}} - gh}$; $1 - \sqrt{1 - c^2}$.

13. (a) Min. at $x = \frac{b}{\sqrt[3]{2}}$; point of inflexion at $x = -b$.

(b) Max. at $x = 0$; points of inflexion at $x = \pm \frac{a}{\sqrt{2}}$.

14. $20\sqrt{2}$. **16.** (a) $\frac{a^4b^4v^2}{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}$. (b) $-\frac{12}{5^{\frac{3}{2}}}$.

17. (a) $\frac{4}{3} a \sin 2\theta$. (b) $8 b \sin \frac{3}{2}\theta$.

Art. 76. Page 173.

9. $\log \sqrt{3}$.

10. $\sqrt{2} - 1$.

11. $\frac{1}{2}$.

12. 1.

13. 2.

14. $r \arcsin \frac{r}{a}$.

15. $\frac{3}{4}(a^{\frac{2}{3}} - 1)^2$.

16. 2.

17. $\frac{\pi}{4}$.

18. $\frac{10\sqrt{10} - 1}{54}$.

Art. 78. Pages 175-176.

1. $\frac{1}{2}$ arc tan 4.

2. $\frac{\pi}{6}$.

3. $\frac{1}{3}$.

4. $\frac{1}{2}(e^4 - 1)$.

5. $\arctan e - \frac{\pi}{4}$.

6. $\frac{1}{2}$.

7. $\frac{1}{3}\log^3 2$.

8. $\frac{1}{a^3\sqrt{2}}$.

Art. 83. Page 187.

1. $\frac{\pi}{2a}$.

2. $\frac{1}{a}$.

3. ∞.

4. $\frac{\pi}{2}$.

8. $\frac{1}{2}(1 + \sqrt[3]{15})$.

Miscellaneous Exercises. Pages 187-189.

1. (a) $\frac{2}{3}\sqrt{2gs_1^3}$. (b) $\frac{4}{3}a^3$. (c) $\frac{2}{15}$.

3. (a) $\frac{1}{2}(\cos\alpha + \cos\beta)$. (b) $\frac{2}{3}a$. **4.** $e^4 - e$.

5. $\frac{kbh^2}{2}$. **6.** (a) $\int_0^{\pm 4}$ (b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}$. (c) $\int_0^{\pm 8}$. (d) $\int_0^{\frac{\pi}{2}}$.

8. (a) $2\sqrt{2}$. (b) $\frac{\pi}{2}$. (c) ∞. **13.** 42.5 and 31.25.

14. (a) 1.5 (b) $\begin{cases} \text{max. if } x = \frac{3 - \sqrt{17}}{2}. \\ \text{min. if } x = \frac{3 + \sqrt{17}}{2}. \end{cases}$ (c) arc tan $\frac{8}{7}$.

15. $5\sqrt{5}$; $\frac{1}{3}\sqrt{5}$.

Art. 85. Pages 193-194.

1. 78.

2. $106\frac{2}{3}$.

3. 10.

4. 32.

6. $a \sinh \frac{m}{a}$.

7. $\frac{m\sqrt{C}}{m-1}\left(b^{\frac{m-1}{m}} - a^{\frac{m-1}{m}}\right); C \log \frac{b}{a}$

8. $\frac{4\pi^2 a^2}{3}$.

9. a^2.

10. $4\pi^2 a^2$.

Art. 86. Page 196.

1. (a) $\frac{1}{3}\pi n^2 m$. (b) $\frac{1}{3}\pi m^2 n$.

2. 27π.

3. 144π.

4. π.

5. $\frac{1}{4}\pi a^2(e^c + 2c - e^{-c})$.

Art. 87. Page 198.

3. 108π.

4. $\frac{4}{3}\pi abc$.

5. $a^2 h(\pi - \frac{4}{3})$.

Art. 88. Page 200.

1. $6a$.

2. $\frac{8a}{27}\left[\left(1 + \frac{9m}{4a}\right)^{\frac{3}{2}} - 1\right]$.

3. $\frac{a}{2}\left(e^{\frac{x_1}{a}} - e^{\frac{-x_1}{a}}\right)$.

4. $2\pi r$.

5. $\frac{a\pi^2}{2}$.

Art. 89. Pages 201-202.

1. $2\pi a$.

2. $\frac{\sqrt{1+a^2}}{a}\left(e^{\frac{\pi a}{2}} - 1\right)$; $\frac{\sqrt{1+a^2}}{a}\left(e^{\pi a} - e^{\frac{\pi a}{2}}\right)$.

3. $\frac{\sqrt{1+a^2}}{a}$.

4. a.

Art. 90. Page 204.

1. $\frac{32\pi}{3}(2\sqrt{2} - 1)$; $\frac{32\pi}{3}(5\sqrt{5} - 2\sqrt{2})$.

2. $\pi n\sqrt{m^2 + n^2}$.

3. $\frac{2\pi}{27}[(37)^{\frac{3}{2}} - 1]$.

4. $\frac{\pi a^2}{4}(e^2 - e^{-2} + 4)$.

Art. 91. Page 206.

1. $\frac{2}{3}\sqrt{5}$.

2. $-5\frac{2}{3}$.

3. (a) $\frac{2}{\pi}$. (b) 0.

4. $\frac{C}{v_a}\left(1 + \log\frac{v_a}{v_b}\right)$.

5. 34.601 lb./sq. in.

Art. 93. Page 210.

1. $4303\frac{1}{3}$ in.-lb.

2. $Mh + \frac{mh^2}{2}$.

4. $W = \frac{k}{3}(s_2^3 - s_1^3)$.

5. 83,219 ft.-lb.

6. 65,684 ft.-lb.

7. 3,273,000 in.-lb.

Miscellaneous Exercises. Pages 211-212.

1. (a) 372. (b) 48. (c) $e-1$.

2. $\dfrac{\pi a^2}{8}$.

3. log 2.

4. $\dfrac{a^2}{6}$.

5. (a) $\frac{32}{15}\pi$. (b) $\frac{4}{3}\pi$

6. $\dfrac{\pi a^3}{15}$.

7. 128 cu. in.

9. $\dfrac{\pi}{2}$.

10. $\frac{2}{3}a^3$.

11. a^2.

12. 144; $\dfrac{(73)^{\frac{3}{2}}-27}{16}$.

13. $\frac{2}{3}mn^2$.

14. $\dfrac{2i_0}{\pi}$.

15. $\dfrac{2p_0}{\pi}$.

16. $C\log\dfrac{v_2-b}{v_1-b}-\dfrac{a(v_2-v_1)}{v_1v_2}$.

17. $\dfrac{k(s_2^{n+1}-s_1^{n+1})}{n+1}$.

18. 56,530 ft.-lb.

Art. 94. Page 215.

1. $\log\dfrac{(x-3)^5}{(x-2)^4}+C$.

2. $\log\sqrt{(x+5)^5(x-3)}+C$.

3. $\log\dfrac{x^{\frac{1}{3}}(x-4)^4}{(x-3)^{\frac{13}{3}}}+C$.

4. $\log\dfrac{(x+2)^5}{(x+1)^3}+C$.

5. $\log\sqrt[5]{\dfrac{3x+1}{x+2}}+C$.

6. $\frac{1}{2}x^2-x-\frac{4}{5}\log x+\frac{13}{15}\log(x-5)-\frac{2}{3}\log(x+1)+C$.

7. $\log\left[\dfrac{(x+m)(x-n)}{x}\right]+C$.

8. $\log[x^2(x-2)^{\frac{3}{2}}(x+2)^{\frac{3}{2}}]+C$.

Art. 94. Page 216.

1. $\log\dfrac{x-2}{x}+\dfrac{1}{x-2}+C$.

2. $\log\sqrt[9]{\left(\dfrac{x+3}{x}\right)^2}-\dfrac{11}{3(x+3)}+C$.

3. $\dfrac{9-4x}{2(x-2)^2}+C$.

4. $\log(x+3)+\dfrac{4}{x+3}+C$.

5. $\log\dfrac{y-3}{y}-\dfrac{1}{y}+C$.

6. $\log\dfrac{(x-1)^8}{x^3}-\dfrac{1}{x}+\dfrac{5}{x-1}+C$.

7. $\log\dfrac{x-1}{x+1}-\dfrac{1}{x-1}+C$.

8. $\dfrac{1}{4}\log\dfrac{x-1}{x+1}-\dfrac{1}{2(x+1)}+C$.

9. $\log x-\dfrac{3}{x}+C$.

10. $m\log(m+x)+\dfrac{2m^2}{m+x}-\dfrac{m^3}{2(m+x)^2}+C$.

Art. 94. Page 217.

1. $\log \dfrac{(x-1)^3(x+1)^2}{(x^2+1)^{\frac{3}{2}}} + \text{arc } 4 \tan x + C.$

2. $\frac{1}{4}\log\dfrac{x-1}{x+1} - \frac{1}{2}\text{arc tan } x + C.$

4. $\frac{1}{4}\log\dfrac{x^2+1}{x^2+3} + C.$

3. $\dfrac{5 \text{ arc tan } x}{2} - \dfrac{x}{2(1+x^2)} + \dfrac{1}{x} + C.$

5. $\frac{1}{6}\log\dfrac{x^2+1}{x^2+4} + C.$

6. $-\dfrac{z}{z^2+1}.$

7. $\frac{3}{8}\log\dfrac{y^2+3}{y-1} + \frac{5}{8}\log(y+1) + \dfrac{1}{\sqrt{3}}\text{arc tan }\dfrac{y}{\sqrt{3}} + C.$

8. $\dfrac{1}{a^2+b^2}\left[\log\dfrac{x+a}{\sqrt{x^2+b^2}} + \dfrac{a}{b}\text{ arc tan }\dfrac{x}{b}\right] + C.$

9. $-\frac{1}{3}\left[\dfrac{1}{x} + \dfrac{1}{\sqrt{3}}\text{arc tan }\dfrac{x}{\sqrt{3}}\right] + C.$

10. $\log\dfrac{\sqrt[4]{x^2+1}}{\sqrt{x+1}} + \dfrac{\text{arc tan } x}{2} + C.$

11. $\log\dfrac{(1+x+x^2)^{\frac{3}{2}}}{(1+x)^3} - \dfrac{1}{1+x} - \dfrac{1}{\sqrt{3}}\text{arc tan }\dfrac{2x+1}{\sqrt{3}} + C.$

12. $\log\dfrac{x}{\sqrt{1+2x^2}} + C.$

13. $\log\sqrt{x^2+1} + \dfrac{1}{2(x^2+1)} + C.$

Art. 95. Page 220.

1. $3\left[\dfrac{x^{\frac{5}{3}}}{5} - \dfrac{x^{\frac{4}{3}}}{4} + \dfrac{x}{3} - \dfrac{x^{\frac{2}{3}}}{2} + x^{\frac{1}{3}} - \log(x^{\frac{1}{3}}+1)\right] + C.$

2. $-\left[\dfrac{4x^{\frac{5}{6}}}{5} + \dfrac{4x^{\frac{2}{3}}}{3} + \dfrac{64x^{\frac{1}{2}}}{27} + \dfrac{128x^{\frac{1}{3}}}{27} + \dfrac{1024x^{\frac{1}{6}}}{81} + \dfrac{4096}{243}\log(3x^{\frac{1}{6}}-4)\right] + C.$

3. $\dfrac{2}{m^3}\left[\dfrac{(mx+b)^{\frac{7}{2}}}{7} - \dfrac{2b(mx+b)^{\frac{5}{2}}}{5} + \dfrac{b^2(mx+b)^{\frac{3}{2}}}{3}\right] + C.$

4. $2\text{ arc tan}\sqrt{\dfrac{x-1}{3-x}} + C.$

6. $\log\left[x + \frac{5}{2} + \sqrt{x^2+5x-3}\right] + C.$

7. $\log\left[x - \frac{7}{2} + \sqrt{x^2-7x+4}\right] + C.$

5. $6\text{ arc tan}\sqrt{\dfrac{x-2}{5-x}} + C.$

8. $\sqrt{\dfrac{2}{3}}\text{ arc tan}\sqrt{\dfrac{3(x-2)}{2(3-x)}} + C.$

9. $2\left\{\dfrac{(x-1)^{\frac{7}{2}}}{7} + \dfrac{3(x-1)^{\frac{5}{2}}}{5} + (x-1)^{\frac{3}{2}} + (x-1)^{\frac{1}{2}}\right\} + C.$

10. $3\sqrt{x^2-3} - \log\left[x + \sqrt{x^2-3}\right] + C.$

11. $\log\left[x + 1 + \sqrt{x^2+2x+5}\right] + C.$

12. $\dfrac{(1-x^2)^{\frac{3}{2}}}{3} - (1-x^2)^{\frac{1}{2}} + C.$

14. $-\dfrac{2\sqrt{2x-x^2}}{x} - \text{arc vers } x + C.$

13. $\dfrac{x^2+2}{\sqrt{x^2+1}} + C.$

15. $\log\left[x + \frac{1}{2} + \sqrt{x^2+x}\right] + C.$

16. $3(x-1)^{\frac{2}{3}} + 6(x-1)^{\frac{1}{3}} + 6\log\left[(x-1)^{\frac{1}{3}} - 1\right] + C.$

Miscellaneous Exercises. Pages 211-212.

1. (a) 372. (b) 48. (c) $e-1$.

2. $\frac{\pi a^2}{8}$.

3. log 2.

4. $\frac{a^3}{6}$.

5. (a) $\frac{32}{15}\pi$. (b) $\frac{4}{3}\pi$

6. $\frac{\pi a^3}{15}$.

7. 128 cu. in.

9. $\frac{\pi}{2}$.

10. $\frac{2}{3}a^3$.

11. a^2.

12. 144; $\frac{(73)^{\frac{3}{2}}-27}{16}$.

13. $\frac{2}{3}mn^2$.

14. $\frac{2\,i_0}{\pi}$.

15. $\frac{2\,p_0}{\pi}$.

16. $C\log\frac{v_2-b}{v_1-b}-\frac{a\,(v_2-v_1)}{v_1\,v_2}$.

17. $\frac{k\,(s_2^{n+1}-s_1^{n+1})}{n+1}$.

18. 56,530 ft.-lb.

Art. 94. Page 215.

1. $\log\frac{(x-3)^5}{(x-2)^4}+C$.

2. $\log\sqrt{(x+5)^6\,(x-3)}+C$.

3. $\log\frac{x^{\frac{1}{3}}\,(x-4)^4}{(x-3)^{\frac{13}{3}}}+C$.

4. $\log\frac{(x+2)^6}{(x+1)^3}+C$.

5. $\log\sqrt[5]{\frac{3\,x+1}{x+2}}+C$.

6. $\frac{1}{2}x^2-x-\frac{1}{5}\log x+\frac{13}{15}\log(x-5)-\frac{2}{3}\log(x+1)+C$.

7. $\log\left[\frac{(x+m)\,(x-n)}{x}\right]+C$.

8. $\log\left[x^2\,(x-2)^{\frac{5}{8}}\,(x+2)^{\frac{3}{8}}\right]+C$.

Art. 94. Page 216.

1. $\log\frac{x-2}{x}+\frac{1}{x-2}+C$.

2. $\log\sqrt[9]{\left(\frac{x+3}{x}\right)^2}-\frac{11}{3\,(x+3)}+C$.

3. $\frac{9-4\,x}{2\,(x-2)^2}+C$.

4. $\log(x+3)+\frac{4}{x+3}+C$.

5. $\log\frac{y-3}{y}-\frac{1}{y}+C$.

6. $\log\frac{(x-1)^3}{x^2}-\frac{1}{x}+\frac{5}{x-1}+C$.

7. $\log\frac{x-1}{x+1}-\frac{1}{x-1}+C$.

8. $\frac{1}{4}\log\frac{x-1}{x+1}-\frac{1}{2\,(x+1)}+C$.

9. $\log x-\frac{3}{x}+C$.

10. $m\log(m+x)+\frac{2\,m^2}{m+x}-\frac{m^3}{2\,(m+x)^2}+C$.

Art. 94. Page 217.

1. $\log \frac{(x-1)^3 (x+1)^2}{(x^2+1)^{\frac{5}{2}}} + \text{arc } 4 \tan x + C.$

2. $\frac{1}{4} \log \frac{x-1}{x+1} - \frac{1}{2} \text{arc} \tan x + C.$

4. $\frac{1}{4} \log \frac{x^2+1}{x^2+3} + C.$

3. $\frac{5 \text{ arc} \tan x}{2} - \frac{x}{2(1+x^2)} + \frac{1}{x} + C.$

5. $\frac{1}{6} \log \frac{x^2+1}{x^2+4} + C.$

6. $-\frac{z}{z^2+1}.$

7. $\frac{3}{8} \log \frac{y^2+3}{y-1} + \frac{5}{8} \log (y+1) + \frac{1}{\sqrt{3}} \text{arc} \tan \frac{y}{\sqrt{3}} + C.$

8. $\frac{1}{a^2+b^2} \left[\log \frac{x+a}{\sqrt{x^2+b^2}} + \frac{a}{b} \text{ arc} \tan \frac{x}{b} \right] + C.$

9. $-\frac{1}{3} \left[\frac{1}{x} + \frac{1}{\sqrt{3}} \text{arc} \tan \frac{x}{\sqrt{3}} \right] + C.$

10. $\log \frac{\sqrt[4]{x^2+1}}{\sqrt{x+1}} + \frac{\text{arc} \tan x}{2} + C.$

11. $\log \frac{(1+x+x^2)^{\frac{3}{2}}}{(1+x)^3} - \frac{1}{1+x} - \frac{1}{\sqrt{3}} \text{arc} \tan \frac{2x+1}{\sqrt{3}} + C.$

12. $\log \frac{x}{\sqrt{1+2x^2}} + C.$

13. $\log \sqrt{x^2+1} + \frac{1}{2(x^2+1)} + C.$

Art. 95. Page 220.

1. $3 \left[\frac{x^{\frac{5}{3}}}{5} - \frac{x^{\frac{4}{3}}}{4} + \frac{x}{3} - \frac{x^{\frac{2}{3}}}{2} + x^{\frac{1}{3}} - \log (x^{\frac{1}{3}} + 1) \right] + C.$

2. $-\left[\frac{4x^{\frac{5}{6}}}{5} + \frac{4x^{\frac{2}{3}}}{3} + \frac{64x^{\frac{1}{2}}}{27} + \frac{128x^{\frac{1}{3}}}{27} + \frac{1024x^{\frac{1}{6}}}{81} + \frac{4096}{243} \log (3x^{\frac{1}{6}} - 4) \right] + C.$

3. $\frac{2}{m^3} \left[\frac{(mx+b)^{\frac{7}{2}}}{7} - \frac{2b(mx+b)^{\frac{5}{2}}}{5} + \frac{b^2(mx+b)^{\frac{3}{2}}}{3} \right] + C.$

4. $2 \text{ arc} \tan \sqrt{\frac{x-1}{3-x}} + C.$

6. $\log [x + \frac{5}{2} + \sqrt{x^2 + 5x - 3}] + C.$

7. $\log [x - \frac{7}{2} + \sqrt{x^2 - 7x + 4}] + C.$

5. $6 \text{ arc} \tan \sqrt{\frac{x-2}{5-x}} + C.$

8. $\sqrt{\frac{2}{3}} \text{ arc} \tan \sqrt{\frac{3(x-2)}{2(3-x)}} + C.$

9. $2 \left\{ \frac{(x-1)^{\frac{7}{2}}}{7} + \frac{3(x-1)^{\frac{5}{2}}}{5} + (x-1)^{\frac{3}{2}} + (x-1)^{\frac{1}{2}} \right\} + C.$

10. $3\sqrt{x^2-3} - \log [x + \sqrt{x^2-3}] + C.$

11. $\log [x + 1 + \sqrt{x^2 + 2x + 5}] + C.$

12. $\frac{(1-x^2)^{\frac{3}{2}}}{3} - (1-x^2)^{\frac{1}{2}} + C.$

14. $-\frac{2\sqrt{2x-x^2}}{x} - \text{arc vers } x + C.$

13. $\frac{x^2+2}{\sqrt{x^2+1}} + C.$

15. $\log [x + \frac{1}{2} + \sqrt{x^2+x}] + C.$

16. $3(x-1)^{\frac{2}{3}} + 6(x-1)^{\frac{1}{3}} + 6 \log [(x-1)^{\frac{1}{3}} - 1] + C.$

Art. 96. Page 223.

1. $e^x(x^2 - 2x + 2) + C.$

2. $\frac{x^{m+1}}{m+1}\log x - \frac{x^{m+1}}{(m+1)^2} + C.$

3. $\theta \text{ arc sin } \theta + \sqrt{1 - \theta^2} + C.$

4. $\theta \text{ arc cot } \theta + \log\sqrt{1 + \theta^2} + C.$

5. $x \text{ inv. cosh } x \pm \sqrt{x^2 - 1} + C.$

6. $\frac{ae^{ax}\sin bx - be^{ax}\cos bx}{a^2 + b^2} + C.$

7. $\log(1 + \sqrt{2}) - \sqrt{2} + 1.$

8. $\sin\theta(\log\sin\theta - 1) + C.$

9. $\frac{x^3}{3}\text{arc tan } x - \frac{x^2}{6} + \frac{\log(x^2 + 1)}{6} + C.$

10. $\frac{ae^{\frac{x}{a}}}{2}\left(\sin\frac{x}{a} + \cos\frac{x}{a}\right) + C.$

11. $\cos x + x\sin x + C.$

12. $\frac{1}{2}\left[\tan\theta\sec\theta + \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right] + C.$

13. $\frac{a^2}{2}\left[\sqrt{2} + \log(1 + \sqrt{2})\right].$

14. $\frac{2a^5}{15}.$

15. $\frac{x^4}{4}\left[(\log x)^2 - \frac{1}{2}\log x + \frac{1}{8}\right] + C.$

16. $\frac{1}{3}.$

Art. 97. Page 227.

1. $-\left(\frac{x^3}{4} + \frac{3x}{8}\right)\sqrt{1 - x^2} + \frac{3}{8}\text{ arc sin } x + C.$

2. $\frac{x^2 - 2}{3}\sqrt{1 + x^2} + C.$

3. $\frac{1}{4a^5}\left(1 + \frac{3\pi}{8}\right)$

4. $\frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8}\text{ arc sin }\frac{x}{a} + C.$

5. $-\frac{\sqrt{a^2 - x^2}}{a^2x} + C.$

6. $\frac{\pi a^4}{16}.$

7. $\frac{(1 + 2x^2)\sqrt{x^2 - 1}}{3x^3} + C.$

8. $-\frac{x + 3a}{2}\sqrt{2ax - x^2} + \frac{3a^2}{2}\text{ arc vers }\frac{x}{a} + C.$

9. $\frac{(2x - a)\sqrt{x^2 - a^2}}{2a^2x^2} - \frac{1}{2a^3}\text{ arc sec }\frac{x}{a} + C.$

10. $-\frac{(2ax - x^2)^{\frac{3}{2}}}{3ax^3} + C.$

11. $\frac{bx^3 - ax}{8ab(a + bx^2)^2} + \frac{1}{8(ab)^{\frac{3}{2}}}\text{ arc tan }\frac{x\sqrt{b}}{\sqrt{a}} + C.$

12. $\frac{x}{2}\cdot\frac{x^2 + 3a^2}{\sqrt{x^2 + a^2}} - \frac{3a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right) + C.$

Art. 98. Page 231.

1. $\frac{1}{6}\sec^6 x + C.$ 7. $\frac{1}{2}\tan^2 x - \log\sec x + C.$

2. $-\cot x - \frac{1}{3}\cot^3 x + C.$ 8. $\frac{1}{6}\tan^6 x + C.$

3. $-\frac{1}{5}\cot^5 x - \frac{1}{3}\cot^3 x + C.$ 9. $-\sin x - \csc x + C.$

4. $-\frac{1}{3}\cot^3 x + C.$ 10. $\frac{2}{3}\tan^{\frac{3}{2}} x + C.$

5. $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.$ 11. $\frac{3}{4}\sin^{\frac{4}{3}} x - \frac{3}{10}\sin^{\frac{10}{3}} x + C.$

6. $-\frac{1}{2}\cot^2 x + \log\tan x + C.$ 12. $\tan x - \cot x + C.$

Art. 99. Pages 234-235.

1. $\frac{1}{3}\cos^3 x - \cos x + C.$ 3. $\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C.$

2. $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$ 4. $-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{8} - \frac{3}{16}\sin 2\theta + C.$

5. $\frac{1}{6}\cos^5 x\sin x + \frac{5}{24}\cos^3 x\sin x + \frac{5}{16}x + \frac{5}{32}\sin 2x + C.$

6. $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$ 9. $2\sqrt{\tan x} - \frac{2}{3}(\tan x)^{-\frac{3}{2}} + C.$

7. $-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C.$ 10. $\frac{2}{5}(\tan x)^{\frac{5}{2}} + C.$

8. $\frac{1}{8}\theta - \frac{1}{32}\sin 4\theta + C.$ 11. $\frac{\sin^3 x}{2\cos^2 x} + \frac{\sin x}{2} - \frac{1}{2}\log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) + C.$

12. $\sec x + \cos x + C.$ 15. $\tan x + \frac{1}{3}\tan^3 x + C.$

13. $\frac{1}{2}\tan^2 x + \log\tan x + C.$ 16. $\log\tan x - \frac{1}{2}\cot^2 x + C.$

14. $-\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\log\tan\frac{x}{2} + C.$ 17. $-\frac{1}{2}\cot^2 x - \log\sin x.$

18. $\frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) + C.$ 19. $-2\cot 2\theta + C.$

20. $\frac{\sin^5 2\theta}{320} + \frac{1}{256}\left(3\theta - \sin 4\theta + \frac{\sin 8\theta}{8}\right) + C.$

Art. 100. Page 236.

1. $-\frac{\sqrt{a^2 - x^2}}{a^2 x} + C.$ 2. $\frac{x}{\sqrt{a^2 - x^2}} - \text{arc sin}\,\frac{x}{a} + C.$

3. $-\frac{x}{\sqrt{x^2 - a^2}} + \log\left(x + \sqrt{x^2 - a^2}\right) + C.$

4. $\frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + C.$

5. $\frac{x}{8}(2x^2 - 5a^2)\sqrt{x^2 - a^2} + \frac{3a^4}{8}\log\left(x + \sqrt{x^2 - a^2}\right) + C.$

6. $\frac{3}{8}\,\text{arc sin}\,x - \frac{2x^3 + 3x}{8}\sqrt{1 - x^2} + C.$ 9. $\frac{\pi a^2}{4}.$

7. $\frac{1}{3}(x^2 - 2)\sqrt{1 + x^2} + C.$

8. $\frac{\sqrt{x^2 - 1}}{x} - \frac{(x^2 - 1)^{\frac{3}{2}}}{3x^3} + C.$ 10. $\frac{11\sqrt{15}}{64}.$

Miscellaneous Examples. Pages 237-240.

1. (a) $-\frac{1}{\sqrt{1+x^2}}\left(\frac{1}{x}+2x\right)+C.$ (b) $\frac{1}{\sqrt{2}}\text{arc sin}\frac{4x-3}{\sqrt{41}}+C.$

(c) $\frac{3}{13}x^{\frac{13}{3}}-\frac{1}{5}x^{\frac{5}{3}}+C.$ (d) $\frac{4}{3}(\sqrt{x}+a)^{\frac{3}{2}}-4a(\sqrt{x}+a)^{\frac{1}{2}}+C.$

(e) $\frac{(x^2-1)\sqrt{2x^2+1}}{6}+C.$ (f) $(\theta+\epsilon)\cos\epsilon+\sin\epsilon\log\sec(\theta+\epsilon)+C.$

2. (a) $\frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3a^4}{8}\text{arc sin}\frac{x}{a}+C.$

(b) $\frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\text{arc sin}\frac{x}{a}+C.$

4. (a) $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\log\sec x+C.$ (b) $\frac{1}{3}\tan^3 x-\tan x+x+C.$

5. (a) $\log\sqrt[3]{\frac{(x-4)^{10}}{x-1}}+C.$ (b) $\frac{1}{3}x^3\text{arc cos}\,x-\frac{1}{3}\sqrt{1-x^2}+\frac{1}{9}(1-x^2)^{\frac{3}{2}}+C.$

(c) $\frac{1}{2}e^{\text{arc tan}\,x}\frac{1+x}{\sqrt{1+x^2}}+C.$ (d) $\frac{1}{8}e^{2x}(1+2\sin x\cos x+2\cos^2 x)+C.$

6. $\frac{1}{2}\left[\sqrt{a^4-x^4}-a^2\text{arc cos}\frac{x^2}{a^2}\right]+C.$ **8.** (a) $\frac{\pi}{4}.$ (b) $-\frac{4}{9}.$ (c) $\frac{4}{9}.$

9. (a) $\frac{1}{2\sqrt{2}}\log\frac{\sqrt{1+x^2}+\sqrt{2}\,x}{\sqrt{1+x^2}-\sqrt{2}\,x}+C.$

(b) $\frac{1}{2(a^2-b^2)}\log\frac{x^2+b^2}{x^2+a^2}+C.$ (c) $(a+bx^2)^{\frac{3}{2}}\left(\frac{x^2}{5b}-\frac{2a}{15b^2}\right)+C.$

11. (a) $kt=\frac{\log(a-x)-\log a}{(a-b)(a-c)}+\frac{\log(b-x)-\log b}{(a-b)(c-b)}+\frac{\log(c-x)-\log c}{(a-c)(b-c)}.$

(b) $a+\frac{1}{\sqrt{2kt+c}}.$

13. $\pi ab.$ **15.** $3\pi a^2.$ **17.** $\frac{8(b-a)^{\frac{5}{2}}}{15\sqrt{m}}.$

14. $\frac{3}{4}\pi ab.$ **16.** $10\log 10-9.$

18. (a) $\sqrt{e^4+1}+2-\sqrt{2}+\log\frac{1+\sqrt{2}}{1+\sqrt{e^4+1}}.$

(b) $\sqrt{37}-\sqrt{2}+\log\frac{6(1+\sqrt{2})}{1+\sqrt{37}}.$ (c) $2\sqrt{3}.$

19. $\frac{a}{2}\left[\pi\sqrt{1+\pi^2}+\log(\pi+\sqrt{1+\pi^2})\right].$

20. $2a\left[\sqrt{5}-2-\sqrt{3}\log\frac{\sqrt{3}+\sqrt{5}}{2\sqrt{2}+\sqrt{6}}\right].$

21. $\frac{3}{2}\pi a$. **22.** $12a$. **23.** $\frac{4}{3}\pi ab^2$.

24. $5\pi^2a^3$. **25.** $6\pi^3a^3$. **26.** $\frac{1}{2}\pi^2$.

27. $\frac{64}{3}\pi a^3$. **28.** $\frac{1}{2}\pi a^2h$. **29.** $\frac{4}{3}a^2h$.

30. $2\pi b\left[b + \frac{a^2}{\sqrt{a^2-b^2}}\text{ arc cos }\frac{b}{a}\right], 2\pi a\left[a + \frac{b^2}{\sqrt{b^2-a^2}}\text{ arc cos }\frac{a}{b}\right]$.

31. $\frac{128}{5}\pi a^3$. **32.** $\frac{\pi a}{4}$. **34.** $\frac{4a}{\pi}$. **35.** $BT\log\frac{p_1}{p_2} - \frac{C(n-1)}{n}(p_1{}^n - p_2{}^n)$.

Art. 102. Page 243.

1. Discontinuous at (0, 0) with respect to each variable separately and together.

2. At (0, 0) continuous with respect to each variable separately; discontinuous with respect to both taken together.

3. Continuous separately and together.

Art. 103. Page 247.

1. (a) $2xy^5$; $5x^2y^4$. (b) $\cos x\cos y$; $-\sin x\sin y$.

(c) $ye^x + e^y$; $e^x + xe^y$. (d) $15x^4 - 12x^3 + 1$; 0.

(e) $y^x\log y$; xy^{x-1}. (f) $-\frac{y}{x^2}\sec^2\frac{y}{x}$; $\frac{1}{x}\sec^2\frac{y}{x}$.

(g) $y\cos x + \sin y$; $\sin x + x\cos y$.

(h) $-2x\csc^2(x^2-y^2)$; $2y\csc^2(x^2-y^2)$.

2. (a) $5y^2z^{-\frac{1}{2}}$; $10xyz^{-\frac{1}{2}}$; $-\frac{5}{2}xy^2z^{-\frac{3}{2}}$. (b) $e^x\log yz$; $\frac{e^x}{y}$; $\frac{e^x}{z}$.

(c) $\cos x\cos y - \sin x\sin z$;
$\cos y\cos z - \sin x\sin y$;
$\cos x\cos z - \sin y\sin z$.

(d) $\frac{1}{x+y}$; $\frac{1}{x+y}$; $-\frac{1}{z}$.

(e) $(z+y)\cos(yz+zx+xy)$; $(z+x)\cos(yz+zx+xy)$;
$(x+y)\cos(yz+zx+xy)$.

3. $\frac{2\pi rh}{3}$; $\frac{\pi r^2}{3}$. **4.** $\frac{y}{2}$; $\frac{x}{2}$.

6. (a) $2\sec^2(xy^2)\tan(xy^2)\,y^4$; $2y\sec^2(xy^2)[2xy^2\tan(xy^2)+1]$;
$2y\sec^2(xy^2)[1+2xy^2\tan(xy^2)]$;
$2x\sec^2(xy^2)[1+4xy^2\tan(xy^2)]$.

(b) 0; $\frac{1}{y}$; $\frac{1}{y}$; $-\frac{x}{y^2}$.

(c) $m(m-1)x^{m-2}y^n$; $mnx^{m-1}y^{n-1}$; $mnx^{m-1}y^{n-1}$; $n(n-1)x^my^{n-2}$.

(d) $6x$; 1; 1; $-6y$. (e) e^{x+y}; e^{x+y}; e^{x+y}; e^{x+y}.

(f) 0; $-3y^2$; $-3y^2$; $-6xy$.

Art. 105. Page 252.

1. $\frac{1}{6}x^3y^2 + F(x) + \psi(y)$.
2. $-e^x \cos y + F(x) + \psi(y)$.
3. $14387\frac{3}{5}$.
4. $\frac{\pi}{8}$.
5. $a^3\left(2 - \frac{5\pi}{8}\right)$.
6. $\frac{a^2(8 - \pi)}{8}$.
7. $\frac{4}{3}\pi a^4$.
8. $\frac{\pi a^5 c^2 (4 + c^2)}{80}$.
9. $\frac{2}{3}\sqrt{2g}\,(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}})\,b$.
10. $\frac{\pi a^3}{2}$.
11. $\frac{a^3b^3}{15}$.
12. $\frac{4a^{n+2}}{2n + 3}$.

Art. 106. Pages 257-258.

1. (a) $2\cdot(x + \cos x e^{2\sin x})$. (b) $\frac{e^x(x - 1)}{x^2 + e^{2x}}$. (c) $\frac{1}{\sqrt{x^2 + a^2}}$.
   (d) $e^{\frac{1}{x}}\,(2x\cos x - \cos x - x^2 \sin x)$. (e) $e^{ax}\sin x\,(a^2 + 1)$.
2. Increases $3\frac{1}{2}$ sq. units per sec.
3. $\frac{679}{15\sqrt{11}}\,\frac{\text{units}}{\text{sec}}$. 4. $-0.284\,\frac{\text{cu. ft.}}{\text{sec}}$. 5. $\frac{C(1 - n)}{Rv^n}$; $\frac{n - 1}{Rn}\sqrt[n]{\frac{C}{p}}$.

Art. 107. Page 259.

1. (a) $e^xy^2\,dx + 2e^xy\,dy$. (b) $y^x \log y\,dx + xy^{x-1}\,dy$.
   (c) $\cos x \cos y\,dx - \sin x \sin y\,dy$. (d) $\frac{2x\,dx}{y^3} - \frac{3x^2\,dy}{y^4}$.
   (e) $a^xe^y(\log a\,dx + dy)$. (f) $\frac{y\,dx - x\,dy}{x^2 + y^2}$. (g) $2x\,dx + 2y\,dy$.
   (h) $\left(y\,dx - x\,dy\right)\left(\frac{1}{x^2 + y^2} - \frac{1}{\sqrt{x^2 - y^2}}\right)$.
2. (a) $2[(x + yz)\,dx + (y + xz)\,dy + xy\,dz]$.
   (b) $z^{xy}\log z\,(y\,dx + x\,dy) + xy\,z^{xy-1}\,dz$.
   (c) $2\tan\theta\sec^2\theta\tan^2\phi\tan^2\psi\,d\theta$
   $+ 2\tan\phi\tan^2\theta\sec^2\phi\tan^2\psi\,d\phi$
   $+ 2\tan\psi\tan^2\theta\tan^2\phi\sec^2\psi\,d\psi$.
   (d) $\cos x\cos y\cos^2 z\,dx - \sin x\sin y\cos^2 z\,dy - \sin x\cos y\sin 2z\,dz$.
   (e) $\frac{zy\,dx + zx\,dy - xy\,dz}{z^2 + x^2y^2}$. (f) $\frac{2xz\,dx + x^2dz}{a^2 - y} + \frac{x^2z\,dy}{(a^2 - y)^2}$.
3. $dp = R\left(\frac{dT}{v} - \frac{T\,dv}{v^2}\right)$. 4. $dk = C\left(T^n dp + npT^{n-1}dT\right)$.

Art. 108. Page 260.

1. $\dfrac{6\,xy - y^2}{3\,y^2 - 3\,x^2 + 2\,xy}$.
2. $\dfrac{3\,x^2 \sin y + y^3 \sin x}{3\,y^2 \cos x - x^3 \cos y}$.
3. $-\dfrac{y}{nx}$.
4. $\dfrac{pn}{b - v}$.
5. $\dfrac{\rho \sin \theta}{2 \cos \theta - 3\,\rho}$.
6. $-\dfrac{b^2x}{a^2y}$.

Art. 109. Page 262.

1. Approx. 2.6 and 0.26; actual 2.605604 and 0.260056004.
2. Approx. 0.00117; actual 0.00116 +.

4. $\pi\,(a\Delta b + b\Delta a)$.
6. $\dfrac{2\,a\Delta a - \dfrac{b^2 - c^2 + a^2}{b}\Delta b - \dfrac{c^2 - b^2 + a^2}{c}\Delta c}{\sqrt{bc}\,\sqrt{a^2 - (b - c)^2}}$

8. $\frac{1}{2}\,(b\Delta a \sin C + a\Delta b \sin C + ab\Delta C \cos C)$.

Art. 110. Page 267.

1. (b) and (f) are inexact.
5. (a) $k\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)$. (b) $\dfrac{k}{2}(r_1{}^2 - r_2{}^2)$.

Miscellaneous Examples. Pages 268–270.

1. (a) $2\,xy^3z^5$; $3\,x^2y^2z^5$; $5\,x^2y^3z^4$.

   (b) $\dfrac{x^2}{(x^3 + y^3 + z^3)^{\frac{2}{3}}}$; $\dfrac{y^2}{(x^3 + y^3 + z^3)^{\frac{2}{3}}}$; $\dfrac{z^2}{(x^3 + y^3 + z^3)^{\frac{2}{3}}}$.

   (c) $3\,x^2e^{2y} \cos z$; $2\,x^3e^{2y} \cos z$; $-\,x^3e^{2y} \sin z$.

   (d) $\dfrac{z}{(x + y)^2 + z^2}$, $\dfrac{z}{(x + y)^2 + z^2}$, $-\dfrac{x + y}{(x + y)^2 + z^2}$.

2. (a) $\dfrac{4\,y^4 - 3\,x^2y^2}{2\,x^3y - 16\,xy^3 + 15\,y^4}$. (b) $\dfrac{b^2x}{a^2y}$.

   (c) $-\dfrac{ax + hy + e}{by + hx + f}$. (d) $-\dfrac{b^mx^{m-1}}{a^my^{m-1}}$.

3. (a) $\dfrac{x + x^3 + \text{arc tan } x}{(1 + x^2)\sqrt{x^2 + (\text{arc tan } x)^2}}$.

   (b) $3\,x^2e^x \text{ arc sin } x + x^3e^x \text{ arc sin } x + \dfrac{x^3e^x}{\sqrt{1 - x^2}}$.

   (c) $\dfrac{e^{-x}(\sin x - \cos x)}{e^{-2x} + \cos^2 x}$. (d) $e^{ax} \sin x\,(a^2 + 1)$.

4. (a) $2\,xe^{xy} \cos e^{xy} - 2\,xe^{2xy} \sin e^{xy} + x^2e^{xy} \cos e^{xy} - 3\,x^2ye^{2xy} \sin e^{xy}$
   $- x^2ye^{3xy} \cos e^{xy}$;
   $2\,ye^{xy} \cos e^{xy} + xy^2e^{xy} \cos e^{xy} - 2\,ye^{2xy} \sin e^{xy}$
   $- 3\,xy^2e^{2xy} \sin e^{xy} - xy^2e^{3xy} \cos e^{xy}$;
   $x^3\,(e^{xy} \cos e^{xy} - 3\,e^{2xy} \sin e^{xy} - e^{3xy} \cos e^{xy})$.

   (b) 0; $-\,10$; $72\,y$.

   (d) $-\,e^x \cos y$; $-\,e^x \sin y$; $e^x \sin y$.

   (e) $x^2y \sin xy - 2\,x \cos xy$; $xy^2 \sin xy - 2\,y \cos xy$; $x^3 \sin xy$.

7. $\dfrac{-ky^2 - ky\sqrt{x^2+y^2}}{x^3 + xy^2 + xy\sqrt{x^2+y^2}}$; $\dfrac{k}{\sqrt{x^2+y^2}}$ **8.** $\dfrac{B}{v-b}$; $\dfrac{2a}{v^3} - \dfrac{BT}{(v-b)^2}$.

9. $-\dfrac{RT}{v^2}$; $\dfrac{R}{v}$; $\dfrac{R}{p}$. **10.** $\mp \dfrac{20}{3\sqrt{11}}$; $\pm \dfrac{18}{5\sqrt{11}}$.

14. $\dfrac{d^2z}{dx^2} = \dfrac{\partial^2 z}{\partial x^2} + \dfrac{2\partial^2 z}{\partial x\partial y}\dfrac{dy}{dx} + \dfrac{\partial^2 z}{\partial y^2}\left(\dfrac{dy}{dx}\right)^2 + \dfrac{\partial z}{\partial y}\dfrac{d^2y}{dx^2}$; $2\left(1 - 2\dfrac{dy}{dx} - x\dfrac{d^2y}{dx^2}\right)$.

17. $\sqrt{3}$; 20. **18.** $\Delta V = \pi r^2 \Delta h + 2\pi r h \Delta r$; $\dfrac{\Delta h}{h} + \dfrac{2\Delta r}{r}$.

19. $\dfrac{m_1\Delta h_1 + h_1\Delta m_1}{m_2h_2} - \dfrac{m_1h_1(m_2\Delta h_2 + h_2\Delta m_2)}{m_2^2h_2^2}$; $\dfrac{\Delta h_1}{h_1} + \dfrac{\Delta m_1}{m_1} - \dfrac{\Delta h_2}{h_2} - \dfrac{\Delta m_2}{m_2}$.

20. 0.00041241; 0.013142.

22. (a) $\frac{1}{3}xy(x^2+y^2) + \phi(x) + \psi(y)$. (b) $\frac{1}{4}x^4 \sin y + F(x) + f(y)$.

Art. 111. Page 273.

1. bh.

2. $\frac{1}{2}bh$.

3. $21\frac{1}{3}$.

4. $a^2 \operatorname{arc\,sin} \sqrt{1 - \dfrac{b^2}{a^2}} - b\sqrt{a^2 - b^2}$.

5. $\dfrac{a^2}{6}(3\pi - 8)$.

6. 12.

7. 3.549.

8. 0.049.

9. 60.1+.

10. πc^2.

11. πab.

Art. 112. Page 275.

2. $\dfrac{5\pi a^2}{4}$. **3.** $\pi(r_1^2 - r_2^2)$. **4.** $\dfrac{3\pi a^2}{2}$. **5.** $\dfrac{a^2}{2}$. **7.** $1.2146\,a^2$; $2.7854\,a^2$.

Art. 113. Page 278.

4. $\dfrac{4\pi abc}{3}$. **5.** $\dfrac{\pi a^2 c}{2}$. **7.** $\dfrac{4\pi a^3}{35}$. **8.** $\frac{2}{3}a^3 \tan\beta$.

Art. 114. Page 280.

1. $\frac{4}{3}\pi r^3$. **2.** $\dfrac{a^3}{360}$. **3.** $\dfrac{9a^3}{2}$. **4.** $\pi a^3\left(\dfrac{\log(\sqrt{2}+1)}{2\sqrt{2}} - \dfrac{1}{6}\right)$.

Art. 116. Page 284.

1. $axx_0 + byy_0 + czz_0 + d = 0$; $\dfrac{x - x_0}{ax_0} = \dfrac{y - y_0}{by_0} = \dfrac{z - z_0}{cz_0}$.

2. $10x - 9y + 4\sqrt{2}z = 72$; $36(x - 5) = -40(y + 2) = 45(z\sqrt{2} - 1)$.

3. $\operatorname{arc\,cos}\dfrac{10}{\sqrt{213}}$; $\operatorname{arc\,cos}\dfrac{-9}{\sqrt{213}}$; $\operatorname{arc\,cos}\dfrac{4\sqrt{2}}{\sqrt{213}}$.

4. $3x - y - 2z + 6 = 0$; $\operatorname{arc\,cos}\left(-\dfrac{\sqrt{14}}{7}\right)$.

Art. 117. Page 287.

1. $\pi h^2\sqrt{k + k^2}$. **2.** $4r^2m$. **3.** $8a^2$. **4.** 210.74 sq. in.

Miscellaneous Examples. Pages 287-288.

1. (a) 12.022. (b) 0.7478. (c) $\frac{3}{8}\pi ab$. (d) $\frac{3}{2}a^2$.

2. (a) $\frac{1}{3}\pi a^2$. (b) $\frac{a^6(27\sqrt{3}+10\pi)}{64}$. (c) $\frac{8}{3}a^2$.

3. $\frac{\pi abr^3}{c}$.

4. $\frac{8\pi abc}{5}$.

5. $\frac{4\pi m^5}{ab}$.

6. $\frac{8\pi a^3}{3}$.

7. $\frac{2\pi a}{\sqrt{mn}}$.

9. $\frac{m\sqrt{a^2b^2+1}}{a^2}$.

10. $\frac{a^2}{\sin\alpha\cos\alpha}$.

11. $\frac{3}{2}ac+\frac{3}{4}(a^2+4c^2)\arctan\frac{a}{2c}$.

12. $\frac{\pi^2R}{2}(r_2{}^2-r_1{}^2)$.

13. $\frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}$.

14. (a) $\frac{\pi}{2}a^4$. (b) $\frac{a^2b^2}{8}$

15. $\frac{a^2c}{2}\left(\frac{3}{2}+\log\frac{b^2}{ac}\right)$.

Art. 118. Page 290.

1. $\frac{2}{3}ka^3$; $\bar{\gamma}=\frac{4ka}{3\pi}$, if $\gamma=ky$.

2. $\frac{\pi}{3}ka^3$; $\frac{3}{4}ka$.

3. $\frac{1}{2}kl$.

4. $\frac{1}{4}\pi ka^4$; $\frac{3}{5}ka$.

5. $\frac{1}{6}kab^2$; $\frac{1}{3}kb$.

6. $\frac{3}{4}$ of density at base.

Art. 120. Page 296.

1. (a) $\left(\frac{2r}{\pi},0\right)$. (b) $\left(\frac{a[6\sqrt{2}+\log(3-2\sqrt{2})]}{8[\sqrt{2}+\log(\sqrt{2}+1)]},\frac{4a[2\sqrt{2}-1]}{3[\sqrt{2}+\log(\sqrt{2}+1)]}\right)$.

(c) $\left(\frac{2a}{e+1},\frac{a(e^2-4-e^{-2})}{4(e-e^{-1})}\right)$. (d) $\left(\pi a,\frac{4a}{3}\right)$.

2. (a) $\left(0,\frac{4r}{3\pi}\right)$. (b) $\left(\frac{3a}{5},\frac{3a}{4}\right)$. (c) $\left(\frac{2a}{e+1},\frac{a(e^2-4-e^{-2})}{(e-e^{-1})}\right)$.

(d) $\left(\pi a,\frac{5a}{6}\right)$.

3. $\left(\frac{5a}{6},0\right)$.

4. $\left(\frac{a}{5},\frac{a}{5}\right)$.

5. $\left(\frac{5a}{7},0\right)$.

6. $\left(\frac{3\pi a}{16},0\right)$.

7. $\left(\frac{2a}{3},0\right)$.

8. $\left(0,0,\frac{8r}{15}\right)$.

9. $\left(\frac{15\pi a}{256},0,0\right)$.

10. $\bar{x}=\bar{y}=\frac{64a}{35\pi}$.

11. $\bar{x}=\bar{y}=\bar{z}=\frac{2}{5}a$.

13. $1\frac{7}{12}$ in. from base of cylinder.

14. Take mass m_1 at origin and X-axis along the side containing m_1 and m_2; then $\bar{x}=\frac{13}{24}a$, $\bar{y}=\frac{5\sqrt{3}}{24}a$.

Art. 121. Page 298.

1. $\frac{1}{3} bh^3$; $\frac{1}{3} h^2$.
2. $\frac{1}{12} bh^3$; $\frac{1}{12} h^2$.
3. $\frac{1}{2} \pi a^4$; $\frac{1}{2} a^2$.
4. $\frac{2}{5} \pi k a^5$; $\frac{3}{5} a^2$.
5. $\frac{1}{4} bh^3$; $\frac{1}{2} h^2$.
6. $\frac{1}{2} \pi a^3$; $\frac{1}{2} a^2$.
7. $\frac{35}{16} \pi a^4$; $\frac{35}{24} a^2$.
8. $\frac{256}{15} a^3$; $\frac{32}{15} a^2$.
9. $\frac{4}{15} \pi a^5$; $\frac{1}{5} a^2$.
10. $\frac{1}{6} \pi k a^6$; $\frac{1}{3} a^2$.

Art. 122. Page 302.

The squares of the radii of gyration are:

1. (a) $\frac{1}{6} a^2$. (b) $\frac{1}{12} a^2$.
2. (a) $\frac{b^2}{4}$. (b) $\frac{a^2}{4}$. (c) $\frac{5}{4} b^2$. (d) $\frac{a^2 + b^2}{4}$.
3. $\frac{1}{4} (a_1^2 - a_2^2)$.
4. $\frac{2}{5} a^2$.
5. $\frac{3}{10} a^2$.
6. (a) $\frac{1}{2} a^2$. (b) $\frac{3}{2} a^2$. (c) $\frac{a^2}{2} + \frac{h^2}{3}$.
8. $\bar{y} = \frac{b^2 + at - t^2}{2 (a + b - t)}$.
9. (a) $k^2 = \frac{b^3 + (a - t) t^2}{3 (a + b - t)}$. (b) $k_g^2 = k^2 - \bar{y}^2$.

Art. 123. Page 305.

1. $y_c = \frac{4}{5} h$.
2. $y_c = \frac{3}{5} h$.
3. $y_c = \frac{2}{3} \frac{h_2^3 - h_1^3}{h_2^2 - h_1^2}$.
4. $y_c = \frac{3}{4} h$.
5. $y_c = \frac{3 h^2 + 8 hc + 6 c^2}{4 h + 3 c}$.
6. $y_c = \frac{3 \pi}{16} a$.

Art. 124. Page 306.

1. $Q = \frac{2}{15} \frac{b \sqrt{2g}}{h_2 - h_1} [2 h_2^{\frac{5}{2}} - 5 h_2 h_1^{\frac{3}{2}} + 3 h_1^{\frac{5}{2}}]$.
2. $\frac{2}{15} \frac{\sqrt{2 g}}{h_2 - h_1} [3 (b_2 h_2^{\frac{5}{2}} + b_1 h_1^{\frac{5}{2}}) - 5 (b_2 h_2^{\frac{3}{2}} h_1 + b_1 h_1^{\frac{3}{2}} h_2) + 2 (b_2 h_1^{\frac{5}{2}} + b_1 h_2^{\frac{5}{2}})]$.
3. $\frac{4}{15} \sqrt{2 g}\, b h^{\frac{5}{2}}$.

Miscellaneous Exercises. Pages 306-309.

1. $\frac{1}{10}$ of density at vertex.
2. $\frac{5}{8}$ of density at base.
3. $2 \pi^2 a^2 b$; $4 \pi^2 ab$.
4. $\frac{a^2}{\sqrt{3}} \sqrt{\frac{4 a^2}{3} + \pi \left(r + \frac{a}{3}\right)^2}$.
8. 57.08 in.; 57.026 in
9. $k^2 = \frac{b_1 h_1^3 - b_2 h_2^3}{12 (b_1 h_1 - b_2 h_2)}$.
10. 10.814 in.
11. $k\left(1 + \frac{c}{h} \log \frac{c}{c + b}\right)$; $kb \left(h + c \log \frac{c}{c + b}\right)$.
12. $c = \frac{2}{\pi} \left(R + \frac{r_1^2 + r_2^2}{4 R}\right)$.
15. $\frac{8}{3} \pi r^4$.
16. $\bar{x} = \frac{1}{3} OC, \bar{y} = \frac{1}{3} OD, \bar{z} = \frac{1}{3} O A$.

Art. 126. Page 314.

1. Convergent conditionally, $-\infty < x < \infty$.
2. Convergent.
3. Convergent if $x^2 < 1$.
4. Convergent absolutely, $-\infty < x < \infty$.
5. Divergent, $-\infty < x < \infty$.
6. Absolutely convergent, $-\infty < x < \infty$.

Art. 127. Page 317.

1. Converges uniformly.
2. Converges uniformly within the given interval.
3. Does not converge uniformly if $0 < x < 1$.

Art. 132. Page 324.

1. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots, \quad -\infty < x < \infty.$

2. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots, \quad -\infty < x < \infty.$

3. $1 + x \log a + \frac{x^2 (\log a)^2}{2!} + \frac{x^3 (\log a)^3}{3!} + \ldots, \quad -\infty < x < \infty.$

4. $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots, \quad -\infty < x < \infty.$

5. $\log 2 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^4}{2^3 \cdot 4!} \ldots, \quad -\infty < x < \infty.$

6. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots, \quad -1 < x \leqq 1.$

7. $x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \ldots, \quad -1 < x < 1.$

8. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots, \quad -1 \leqq x \leqq 1.$

9. $1 - n \frac{\theta^2}{2!} + (3n^2 - 2n) \frac{\theta^4}{4!} - \ldots, \quad -\infty < \theta < \infty.$

10. $1 + \theta + \theta^2 + \frac{2\theta^3}{3} + \frac{\theta^4}{2} + \ldots, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$

11. $1 + \frac{\theta^2}{2} + \frac{5\theta^4}{24} + \frac{61\theta^6}{720} + \frac{277\theta^8}{8064} + \ldots, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$

12. $1 + \theta + \frac{\theta^2}{2!} + \frac{2\theta^3}{3!} + \frac{5\theta^4}{4!} + \ldots, \quad -1 \leqq \theta \leqq 1.$

13. $x - \frac{x^3}{3!} + \frac{9x^5}{5!} - \ldots, \quad -\infty < x < \infty.$

14. $-\left(\frac{\theta^2}{2}+\frac{\theta^4}{12}+\frac{\theta^6}{45}+\frac{17\,\theta^8}{2520}+\ldots\right),\quad 0\leqq\theta<\frac{\pi}{2}.$

15. $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots,\quad -\infty<x<\infty.$

16. $-x+\frac{x^2}{2}+\frac{2\,x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}+\ldots,\quad -\infty<x<\infty.$

Art. 133. Page 327.

1. $e^x\left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\ldots\right).$

2. $x^m+mx^{m-1}y+\frac{m\,(m-1)}{2!}x^{m-2}\,y^2+\ldots$

3. $x^6+6\,x^5y+15\,x^4y^2+20\,x^3y^3+15\,x^2y^4+6\,xy^5+y^6.$

4. $\text{arc}\sin x+\frac{h}{(1-x^2)^{\frac{1}{2}}}+\frac{h^2x}{2!\,(1-x^2)^{\frac{3}{2}}}+\frac{h^3\,(1+2\,x^2)}{3!\,(1-x^2)^{\frac{5}{2}}}+\ldots.$

5. $\log\sin x+h\cot x-\frac{h^2}{2}\csc^2x+\frac{h^3}{3}\csc^2x\cot x-\ldots.$

6. $x^3+9\,x^2+23\,x+22.$ 9. $\sinh x\cosh y+\cosh x\sinh y.$

7. $x^2+5\,x-11.$ 10. $\cosh x\cosh y+\sinh x\sinh y.$

11. $\sin a+(x-a)\cos a-\frac{(x-a)^2}{2!}\sin a-\frac{(x-a)^3}{3!}\cos a+\ldots.$

12. $5\,(x-1)^3+9\,(x-1)^2+4\,(x-1)-10;$
$5\,(x-3)^3+39\,(x-3)^2+100\,(x-3)+74.$

13. $(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\ldots,\quad 0<x\leqq 2.$

14. $\frac{1}{a}-\frac{x-a}{a^2}+\frac{(x-a)^2}{a^3}-\frac{(x-a)^3}{a^4}+\ldots,\quad 0<x<2\,a.$

Art. 135. Page 330.

1. $\frac{x^n}{n!}e^{\theta x};\ -\infty<x<\infty.$ 3. $\frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n;-1<x\leqq 1.$

2. $\frac{x^n}{n!}\cos\left(\theta x+\frac{n\pi}{2}\right);-\infty<x<\infty.$ 4. $a^{\theta x}\,(\log a)^n\,\frac{x^n}{n!};\quad -\infty<x<\infty.$

Art. 136. Page 332.

1. $x^3\,(a+y)^2+3\,x\,(a+y^2)\,h^2+6\,x^2\,(a+y)\,hk+x^3k^2$
$+\,(a+y)^2\,h^3+6\,x\,(a+y)\,h^2k+3\,xhk^2+2\,(a+y)\,h^3k+3\,xh^2k^2+h^3k^2.$

2. $f\,(x,y)+h\,(2\,ax+by+d)+k\,(bx+2\,cy+g)+ah^2+bhk+ck^2.$

3. $y^n\,(x+h)+nkxy^{n-1}.$

4. $e^x\sin y\left[1+h+\frac{h^2}{2}+\frac{h^3}{3!}+\ldots\right]\left[1-\frac{k^2}{2}+\frac{k^4}{4!}-\ldots\right]$

$+e^x\cos y\left[1+h+\frac{h^2}{2}+\frac{h^3}{3!}+\ldots\right]\left[k-\frac{k^3}{3!}+\frac{k^5}{5!}-\ldots\right].$

Art. 137. Page 333.

2. $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots, \quad |x| < 1.$

3. $x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \dots$

Art. 138. Pages 338-339.

1. 0.25882, 0.96593.

2. 0.21256.

3. 11.008.

4. 3.433987. 4.290459.

5. 0.009804; 0.010309.

6. $\frac{r}{8 \cdot 35^{\frac{7}{8}}}.$

7. 0.016 +.

8. $-19° 15' < x < 19° 15'.$

9. Error $< \frac{m^2}{2!} + \frac{m^3}{3!}.$

10. 0.54930.

11. (a) 10.472. (b) 10.50. Relative error = 0.0027.

12. 5236 ft.; 0.0002.

13. 2.0033 in.; 0.0001.

14. $\frac{2h^2}{c}\left(1 - \frac{h^2}{c^2}\right).$

Art. 139. Page 341.

1. Min. if $x = \frac{\pi}{4} + n\pi.$

2. No max. nor min.

3. Max. if $x = \frac{\pi}{3}$; min. if $x = -\frac{\pi}{3}.$

4. Min. if $x = \frac{1}{2a} \log \frac{q}{p}.$

5. Max. if $u = 0.$

6. Max. if $x = -\tan x, \frac{\pi}{2} < x < \pi.$

Min. if $x = -\tan x, \frac{3\pi}{2} < x < 2\pi.$

7. Max. if $\theta = 0$; min. if $\theta = \pi$, or arc cos $\sqrt[3]{\frac{2}{3}}.$

8. Min. if $x = 0.$ **9.** Min. if $x = 1.$ **10.** Min. if $x = 0.$

Art. 140. Pages 344-345.

1. 2. **2.** $\frac{3}{10}$. **3.** $\frac{1}{2}$. **4.** $\log \frac{a}{b}$. **5.** 0. **6.** 0.

7. -1. **8.** 2. **9.** -2. **10.** $\frac{1}{3}$. **11.** $\log a$. **12.** ∞.

13. 0. **14.** $\frac{\pi^2}{8}$. **15.** 0. **16.** 1. **17.** 1. **18.** 1.

19. e^2. **20.** 0. **21.** $\frac{1}{3}$. **22.** $-\frac{1}{3}$. **23.** 1. **24.** ∞.

25. $\frac{7}{8}$. **26.** $\frac{1}{2}$.

Miscellaneous Exercises. Pages 345-347.

2. (a) Convergent if $x < \frac{3}{2}$. (b) Convergent if $-\infty < x < \infty$.

(c) Convergent if $-\infty < x < \infty$. (d) Convergent if $x \neq 0$.

3. (a) $\theta + \frac{\theta^3}{3} + \frac{2\,\theta^5}{15} + \frac{17\,\theta^7}{315} + \dots$

(b) $1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{8\,x^7}{7!} + \dots$

(c) $1 - \frac{nx^2}{2!} + (3\,n^2 - 2\,n)\frac{x^4}{4!} - \dots$

(d) $1 - \frac{x}{2} + \frac{x^2}{12} + \dots$ (e) $2 + \frac{x^2}{4} + \frac{x^4}{96} + \frac{x^6}{1440} + \dots$

5. $x + \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} + \frac{x^7}{7 \cdot 7!} + \dots$ **7.** 2.018.

8. $275 \pm 85\,h + h^2 \pm h^3$;
249.617; 258.048; 266.511; 283.509; 292.032; 300.563.

11. $\frac{4}{3}\,b^2\left[\frac{2\,bh}{b^2 + h^2} + \frac{3}{16}\left(\frac{2\,bh}{b^2 + h^2}\right)^3 + \dots\right]$.

12. $x + \frac{2}{3}\,x^3 + \frac{8}{15}\,x^3 + \frac{16}{35}\,x^7 + \dots$

15. (a) 1. (b) $\frac{1}{6}$. (c) 2.

16. (a) 0.17365; 0.98481.
(b) 2.833213; 3.433987; 4.110874.
(c) 2.9994.

18. (a) $1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3\,x^4}{8} \dots$ (b) $x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{3\,x^5}{40} + \dots$

(c) $1 \pm x - \frac{x^2}{2} \mp \frac{x^3}{6} + \frac{x^4}{24} \pm \dots$ (d) $1 - x^2 + \frac{x^4}{3} - \frac{2\,x^6}{45} + \dots$

Art. 142. Page 352.

1. $y = 3\,x - 12$. **3.** $4\,x + y = 12$.

2. $(x + 54)^2 + (y - 26)^2 = 4000$. **4.** $(x - 5)^2 + (y + 2)^2 = 8$.

5. $\left(x - \frac{a^2 - b^2}{a}\right)^2 + y^2 = \frac{b^4}{a^2}$; $x^2 + \left(y - \frac{b^2 - a^2}{b}\right)^2 = \frac{a^4}{b^2}$.

Art. 144. Page 357.

1. The lines $x = -2$, $x = 8$. **3.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.

2. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = c^{\frac{1}{2}}$. **4.** $xy = \pm \frac{1}{2}\,k^2$.

5. $(x - 2\,a)^3 = \frac{27}{4}\,ay^2$.

Art. 147. Page 362.

1. $4(m-2p)^3 = 27pn^2$.
2. $(am)^{\frac{2}{3}} + (bn)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$.
3. The origin.
4. $(m+n)^{\frac{2}{3}} + (m-n)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.
5. The catenary, $n = \frac{a}{2}\left(e^{\frac{m}{a}} + e^{-\frac{m}{a}}\right)$.

Art. 148. Page 366.

1. $y = x$.
2. $x = a,\ y = \pm(x+a)$.
3. $x = 2a$.
4. $y = \pm\frac{b}{a}x$.
5. $x = 2$.
6. $3x + 3y = 2$.
7. $x = \pm c;\ y = \pm c$.
8. $x = 1$.
9. $x = a;\ y = x + \frac{a}{2};\ y = -x - \frac{a}{2}$.
10. $x = 2a$.
11. $y = x + 2$.
12. $x = 0;\ y = 0$.
13. $y = x + \frac{1}{3}$.
14. $x = 2;\ y = -2;\ x + y = 3$.
15. $x = \pm 2;\ y = 0;\ y = x$.
16. $x + y + a = 0$.
17. $y = 0;\ y = \pm x - 1$.
18. $y = \pm x$.
19. $xy = 0;\ x + y = 0$.
20. $x = -1$.
21. None.

Art. 149. Page 367.

1. $\rho\sin\left(\text{arc}\cos\frac{1}{c} \pm \theta\right) = \frac{a(1-e^2)}{\sqrt{c^2-1}}$.
2. $\theta = 0$.
3. $\rho\sin\theta = 2\pi$.
4. $\rho\sin\left(\frac{\pi}{4} - \theta\right) = a;\ \rho\sin\left(\frac{5\pi}{4} - \theta\right) = a$.
5. $\rho\sin\theta = \pm\frac{a}{2};\ \rho\cos\theta = \pm\frac{a}{2}$.
6. $\rho\sin\theta = b$.
7. $\rho\cos\theta = 2a$.
8. No asymptote.
9. $\rho\sin\theta = \pm\frac{a}{2};\ \rho\cos\theta = \pm\frac{a}{2}$.
10. $\rho\sin\theta = \pm\frac{a}{2};\ \rho\cos\theta = \pm\frac{a}{2}$.

Art. 151. Page 375.

1. Node at (0, 0); $y = \pm x$.
2. (0, 0) is a conjugate point.
3. Cusp of first species at (0, 0); $y^2 = 0$.
4. Tacnode at (0, 0); $y^2 = 0$.
5. Triple point at (0, 0); $y = 0,\ y = \pm x$.
6. No singular points.
7. Node at (0, 0); $\sqrt{2}\,y = \pm x$.
8. Cusps of first species at $(\pm a, 0)$ and $(0 \pm a)$; $y^2 = 0,\ x^2 = 0$.
9. Cusp of first species at $(\pm a, 0)$; $y^2 = 0$.
10. Cusp of first species at (0, 0); $y^2 = 0$.
11. Conjugate point at $(-a, 0)$.
12. Conjugate point at $(-b, 0)$.
13. Cusp of second species at (0, 0); $y^2 = 0$.
14. (0, 0) is a singular point. $y = -1.233x$.
15. Tacnode at (0, 0). $y^2 = 0$.

Miscellaneous Exercises. Pages 375-377.

1. (a) $-\frac{(65)^{\frac{3}{2}}}{16}$; $\left(\frac{89}{4}, -\frac{343}{16}\right)$. (b) $-\frac{(785)^{\frac{3}{2}}}{36}$; $\left(\frac{5477}{9}, -\frac{929}{36}\right)$.
(c) -1; $(0, 0)$. (d) $2\sqrt{2}$; $(-2, 3)$.

2. $y + 8x = 17$. 5. $y = \pm\left(x + \frac{a}{2}\right)$; $x = a$.

9. $x + 1 = 0$; $y = x^2 - x + 1$.

11. (a) Node at $(a, 0)$; $y = \pm(x - a)$.
(b) Conjugate point at $(0, 0)$.
(c) Node at $(0, 0)$; $y = 0$, $4y = 3x$.

12. (a) $y = x + \frac{4}{3}$. (b) $y = \pm\frac{b}{a}x$.

15. $y = \pm x$; $y - b = \pm\sqrt{a}(x - a)$.

16. $\left(\frac{m \log b - 2mn}{2n^2}, ab^{\frac{\log b - 2n}{n \log b}}\right)$; $\tau = -\frac{m}{n}$; $p = ab^{\frac{1}{n}}$.

17. $2RTv^4 - 6a(v - b)^3 = 0$; $v = 0.2299 +$, or $0.00297 +$.

18. $\frac{a}{v^2} - \frac{RTb}{(v - b)^2} = 0$, $\frac{RTb}{(v - b)^3} - \frac{a}{v^3} > 0$.

19. $0; \left(\pm\frac{\sqrt{2}}{2a}, \frac{k}{\sqrt{e}}\right)$. 20. (a) (0.040115, 0.77192). (b) 34.303.

21. $\frac{4}{3}\sqrt{2 + \sqrt{3}}$.

24. $4096a^3m + 1152a^2n^2 + 27n^4 = 0$. 25. $x^3 + y^3 = p^3$.

Art. 152. Page 380.

1. 0.523599. 4. $2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4}\right)$.

2. 1.31105. 5. $\frac{303}{256}x + \frac{47}{512}\sin 2x + \frac{5}{128}\sin x \cos^3 x + \ldots$

3. $5.244a$. 6. 0.746824.

Art. 154. Page 387.

1. 2286; 2286; 2289; 2285.6. 4. 41610 ft. lb.

2. 0.69315; 0.69312. 5. $\frac{h}{3}(r_1^2 + r_1r_2 + r_2^2)$.

3. 0.5000075.

Miscellaneous Exercises. Pages 387-388.

1. (a) 0.319. (b) 9.798. (c) 1.2527.

4. (a) $-a^2\left[\frac{c}{a}\log\frac{2c + a}{2c - a} - 1\right]$.

(b) $-a^2\left[\frac{1}{3}\left(\frac{a}{2c}\right)^2 + \frac{1}{5}\left(\frac{a}{2c}\right)^4 + \frac{1}{7}\left(\frac{a}{2c}\right)^6 + \ldots\right]$.

5. $\pi a^2\sqrt{2gh}\left[1 - \frac{1}{32}\frac{a^2}{h^2} - \frac{5}{1024}\frac{a^4}{h^4} - \frac{105}{65536}\frac{a^6}{h^6} - \ldots\right]$.

Art. 156. Page 391.

1. (a) $\frac{d^2y}{dx^2} = 0$. (b) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y = 0$. (c) $\frac{dy}{dx} = \cot x$.

2. $\frac{d^2s}{dt^2} = g$. 3. (a) $\frac{d^2y}{dx^2} - m^2x = 0$. (b) $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y = 0$.

4. $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$.

Art. 158. Page 394.

1. (a) $3y^4 = 4x^3 + C$. (b) $y = b + C(a - x)$. (c) $y^2 = 2\arcsin x + C$.
(d) $y = Ce^{\frac{1}{x}} + 2$. (e) $pv = C$. (f) $\tan x - \sec y = C$.

2. $y = Cx^{\frac{1}{n}}$.
3. $y = e^x$.
5. $v^2 = \frac{2k}{x} + C$.

Art. 159. Page 395.

1. $(y + x)^2 = cy$.
2. $x^2 - 2y^2 = cx^{\frac{3}{2}}$.
3. $x^3 = y^3 \log cy^3$.
4. $xy^2 = c(x + 2y)$.
5. $\log x - \frac{x}{y} = c$.

Art. 160. Page 396.

1. $xy = C$.
2. $y \log x = C$.
3. $xe^y = C$.
4. $ax^2 + 2hxy + 2gx + by^2 + 2fy = C$.
5. $xy + e^x \sin y = C$.
6. $x^3 + 2x^2y - xy^2 + y^3 = C$.
7. (a) $\frac{x}{y} = C$. Integrating factor $\frac{1}{y^2}$.
(b) $x^2y^6 = C$. Integrating factor x^2y^5.
(c) $\log y - \frac{x}{y} = C$. Integrating factor $\frac{1}{y^2}$.

Art. 161. Pages 398-399.

1. $2y\sqrt{1 + x^2} = x\sqrt{1 + x^2} + \log(x + \sqrt{1 + x^2}) + C$.
2. $xy = \sin x - x\cos x + C$.
3. $y = C\sin x - \cos x$.
4. $y = x^n(e^x + C)$.
5. $y = x - 1 + Ce^{-x}$.
6. $xy = \frac{1}{2}e^x + Ce^{-x}$.
8. $\omega = \frac{M}{k} + Ce^{-\frac{kt}{I}}$; $C = \omega_0 - \frac{M}{k}$.

Art. 162. Page 400.

1. $\frac{1}{y^4} = 2x^2 + Cx^4$.
2. $y^4 = \frac{4}{11}x^3 + Cx^{-8}$.
3. $\frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}$.
4. $\sec x = y(\tan x + C)$.
5. $y^{\frac{3}{2}} = \frac{3}{7}(x^2 - 1) + C(x^2 - 1)^{-\frac{3}{4}}$.
6. $y^2 - 1 = Ce^{-x^2}$.

Art. 163. Pages 403-404.

1. $y = c_1e^{-3x} + c_2e^{-2x}$.
3. $y = e^{2x}(c_1 + c_2x) + c_3e^{-x}$.
2. $y = c_1e^{3x} + c_2e^{-5x}$.
4. $y = e^{-x}(c_1 + c_2x + c_3x^2 + c_4x^3)$.
5. $y = c_1e^{x} + e^{-\frac{1}{2}x}(c_2 \cos \frac{1}{2}\sqrt{3}\,x + c_3 \sin \frac{1}{2}\sqrt{3}\,x)$.
6. $y = c_1 \cos 2x + c_2 \sin 2x$.
7. $y = e^{x}(c_1 + c_2x + c_3 \cos 2x + c_4 \sin 2x)$.
8. $y = e^{x}[(c_1 + c_2x) \cos 2x + (c_3 + c_4x) \sin 2x]$.
9. $y = (c_1 + c_2x) \cos \sqrt{2}\,ax + (c_3 + c_4x) \sin \sqrt{2}\,ax$.
10. $y = c_1e^{x} + c_2e^{-x} + c_3$.
11. $y = c_1e^{4x} + c_2e^{-2x} + c_3$.
12. $y = e^{\frac{1}{2}x}(c_1 + c_2x) + c_3e^{-x}$.

Art. 164. Page 406.

1. $y = \frac{1}{24}x^4 + \frac{1}{2}x^3 + \frac{1}{2}c_1x^2 + c_2x + c_3$.
2. $y = e^{x} - \sin x + c_1x + c_2$.
3. $x = \dfrac{1}{2\sqrt{c_1}} \log \dfrac{\sqrt{e^{2y} + c_1} - \sqrt{c_1}}{\sqrt{e^{2y} + c_1} + \sqrt{c_1}} + c_2$.
5. $3(y + c_2) = \log \dfrac{1}{3x + c_1}$.
4. $\dfrac{1}{\sqrt{a}} \log [b + ay + \sqrt{a}\sqrt{ay^2 + 2by + c_1}] = x + c_2$.
6. $\dfrac{1}{1 - y} = c_1x + c_2$.
7. $y = c_1 \arctan x + c_2$.
8. $V = c_1 \log r + c_2$.

Miscellaneous Exercises. Pages 412-413.

1. $(a^2 - x^2)\left(\dfrac{dy}{dx}\right)^2 = x^2$.
2. $y = \dfrac{1}{2k}\left(ce^{kx} + \dfrac{1}{c}e^{-kx}\right)$, a family of catenaries.
3. (a) $y = ce^{-\sqrt{x/y}}$. (b) $e^{y} \sin x = e^{x} + c$. (c) $y = x^2(1 + ce^{\frac{1}{x}})$.
4. $(y - m \log x + c)(y + m \log x - c) = 0$.
6. $\begin{cases} x = c_1e^{-t} - 2c_2e^{-4t}. \\ y = c_1e^{-t} + c_2e^{-4t}. \end{cases}$
7. (a) $y = c_1x^2 + c_2x$. (b) $y = c_1x^2 + c_2x^{3+\sqrt{14}} + c_3x^{3-\sqrt{14}}$.
8. (a) $\dfrac{\pi}{2}\sqrt{\dfrac{a^3}{2k}}$. (b) $v = \sqrt{2k\left(\dfrac{1}{x} - \dfrac{1}{a}\right)}$.
9. $\rho = c_1e^{\omega t} + c_2e^{-\omega t}$, where $\begin{cases} c_1 = \frac{1}{2}\left(a + \dfrac{V_0}{\omega}\right). \\ c_2 = \frac{1}{2}\left(a - \dfrac{V_0}{\omega}\right). \end{cases}$

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